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NOTE ON THE PHOTOGRAPHY OF FABRY AND PEROT INTERFEROMETER FRINGES BY THE USE OF A SIMPLE OPTICAL SYSTEM

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BSTRACT. A simple method of photographing Fabry and Perot interferometer fringes outlined, and some examples of the results obtained with mercury radiation are given.

AFTER some experiments on the photography of the fringe systems produced by a Fabry and Perot interferometer it was thought that a few examples such as those which accompany this note, while possessing no originality, ight not be devoid of interest as roughly indicating the results to be obtained by e use of a simple optical system. The only auxiliary apparatus employed, together ith the necessary camera and source of light, was a convergent lens. The arrangement adopted is shown diagrammatically in Fig. 1.

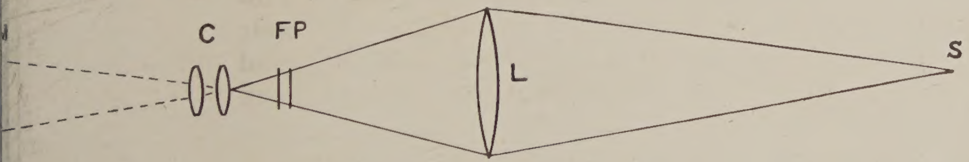

Fig. 1. Arrangement of apparatus.

The condensing lens *L* caused light from the source *S* to pass through the interrometer plates *FP* to form an image of the source approximately at the front of e camera lens *C*, the fringe system being observed on the receiving screen *N*. A nvenient arrangement was obtained by placing the camera lens and the source of ght at distances of about 40 cm. and 80 cm. respectively from the condenser *L*, hich was a single uncorrected lens of approximately 25 cm. mean focal length. his lens was of large aperture (14·5 cm.) and thus possessed the dual advantage of llecting a considerable amount of light from the source while subtending a large lid angle at the camera lens. The interferometer plates were placed as close to the mera lens as the interferometer mounting would allow (about 6 cm. away). By this means the plates, which had an aperture of 2·6 cm., were prevented from cutting off an appreciable fraction of the cone of light converging on to the camera lens from the condenser. Hence the angle of the cone of illumination falling on to the receiving screen (shown approximately by the dotted lines in the diagram) was maintained fairly large. The visible portion of the mercury vapour lamp employed as the source of light being rectangular (3·5 × 2 cm.), the condensing lens *L* was turned until the patch of light on the nearer interferometer plate was approximately square and just about covered the plate so that a minimum quantity of light was lost.

With the optical system arranged approximately as described, the quarter-plate screen of the camera could be almost entirely covered with fringes. The camera lens possessed a focal length of 29 cm. and for the photography was used with the stop at *f*/11.

The use of a medium aperture condensing lens of short focal length, placed closer to the interferometer plates to obtain good covering power on the receiving screen, was prevented by the presence of the interferometer bed which carried the movable plate. This bed extended for a distance of nearly 20 cm. beyond the plates towards the condensing lens. All the apparatus rested on shelves fixed to the laboratory wall, the interferometer being on a separate shelf. Mechanical insulation of the interferometer was further improved by the insertion of a thick layer of cotton wool underneath the baseboard of the instrument.

Ilford Special Rapid Panchromatic plates were employed in taking the photographs shown in Fig. 2. The details were as follows:

Table. Details relating to Fig. 2

Photograph	Fringe system	Approximate plate separation (mm.)	Exposure (sec.)
(*a*)	Full mercury	0·7	0·5
(*b*)	Full mercury	2·0	1·0
(*c*)	Mercury green λ 5461	0·9	75
(*d*)	Mercury green λ 5461	2·8	75

The satellite shown in Fig. 2 (*d*) appears to be situated at about a quarter of an Ångstrom unit from the main line*. Calculation shows that with the plates approximately 3 mm. apart such a satellite should give fringes about equally spaced between those of the principal system. The edges of the main fringes on the sides of higher wave-length are diffuse. These results are in very fair agreement with the measurements given by H. G. Gale and H. B. Lemon in a discussion of the fine structure of the salient mercury lines†. In their paper a grating photograph of λ 5461, obtained from a Cooper-Hewitt lamp, is given.

A filter to isolate this radiation was conveniently made by mixture of dilute solutions of malachite green and potassium bichromate. The glass cell containing

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* This satellite is on the side of lower wave-length.

† *Astrophysical Journal*, **31**, 78 (1910). See also J. C. McLennan, *Proc. R. S.* A. **87**, 276 (1912).

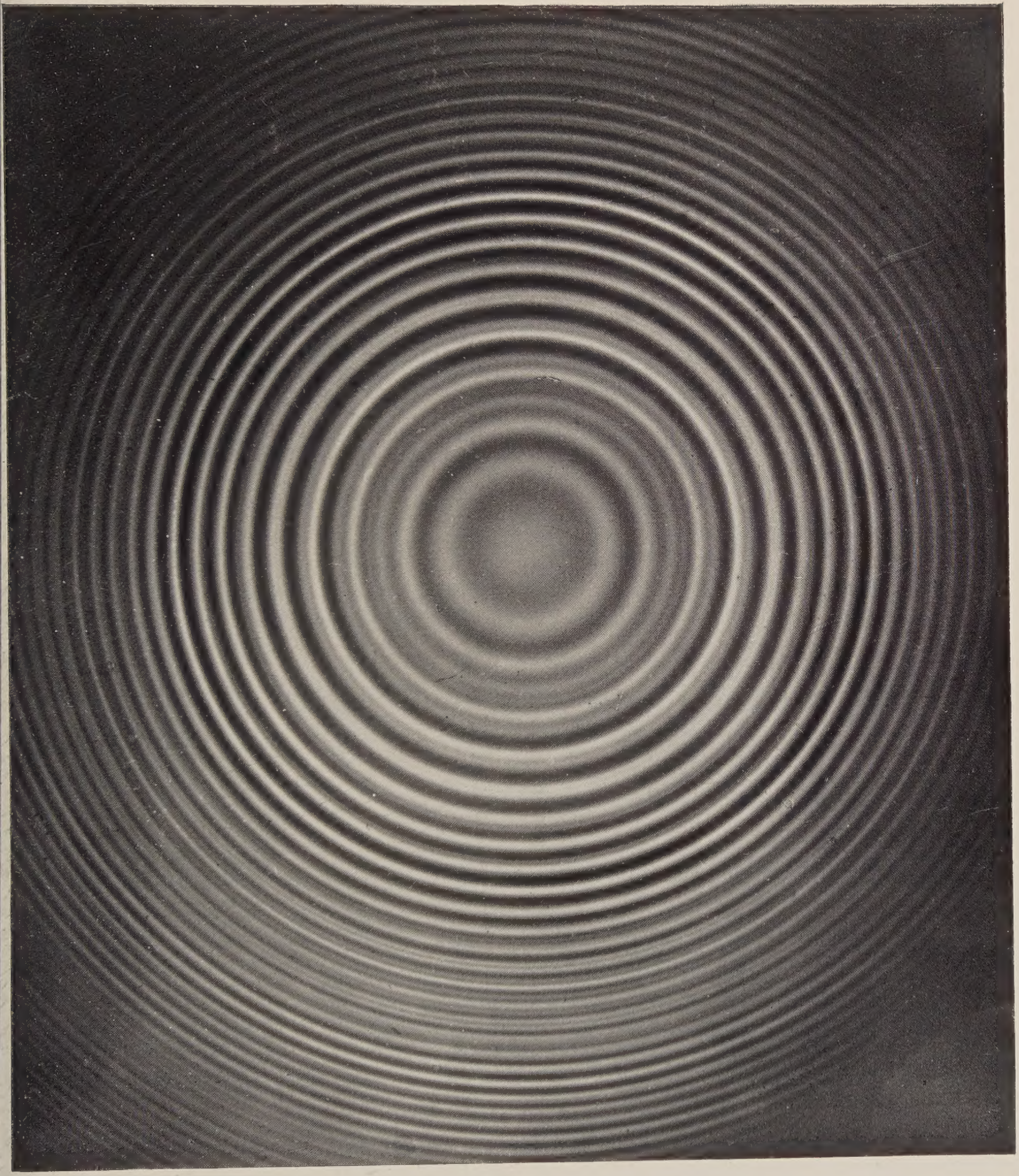

Fig. 2(*a*). (Original width 6·2 cm.)

OLD SYNDICATE ROOM.
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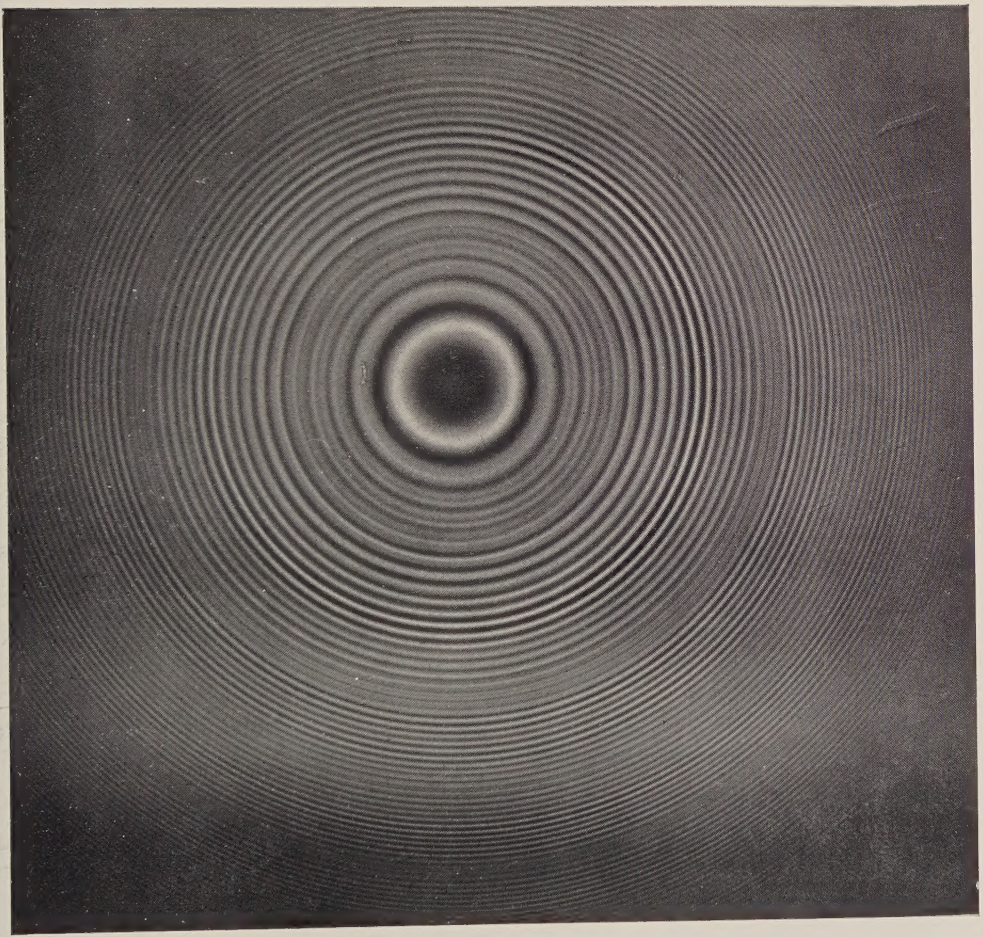

Fig. 2 (*b*). (Original width 5·8 cm.)

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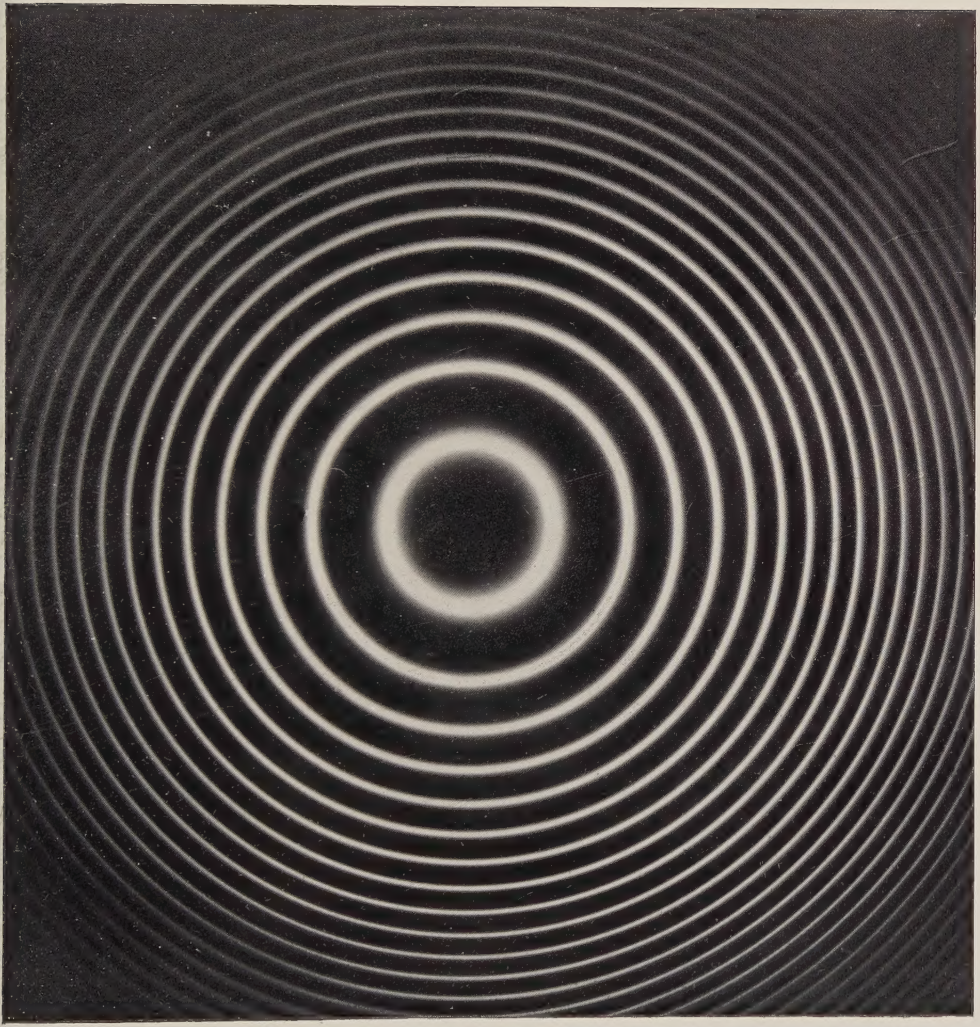

Fig. 2 (*c*). (Original width 5·8 cm.)

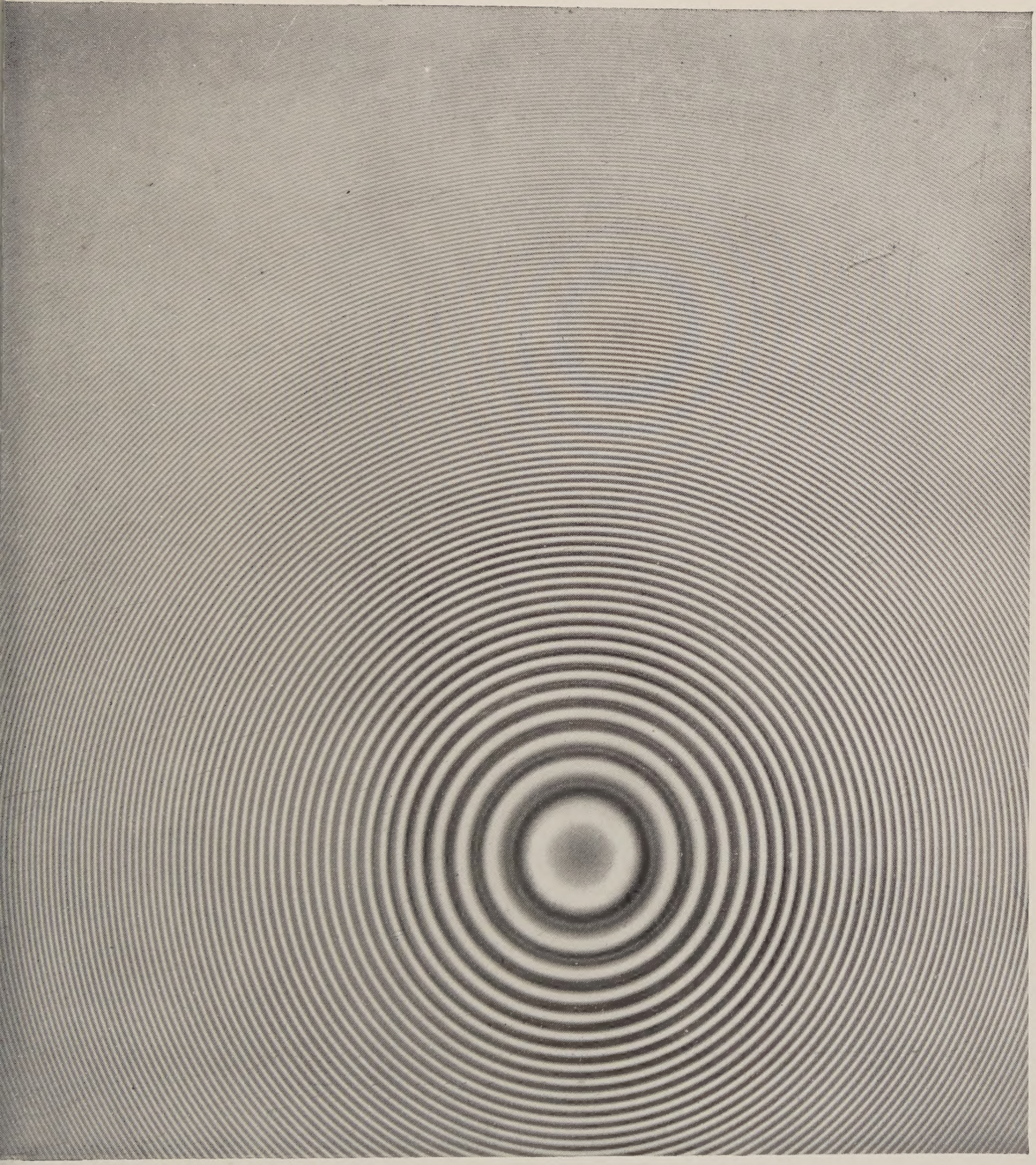

Fig. 2 (*d*). (Original size 4·3 cm.)

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the liquid was placed close to the interferometer plates on the side towards the incident light. The manner in which such a filter may be adjusted to isolate the desired green line is approximately as follows. A fresh dilute solution of malachite green crystals absorbs the orange and red portions of the spectrum roughly between the wave-lengths 6400 and 6000 Å.U. By increase of the concentration this absorption band may be made to extend into the red at 6700 and the green at 5100. The edge of the band in the red is sharp and varies in position with the solution-strength much less than does the more diffuse edge of lower wave-length. When the concentration of the malachite green is adjusted so that the range of elimination stretches from about 6600 to 5500 the strong pair of yellow lines $\lambda\lambda$ 5791, 5770 are cut out while the required green radiation is transmitted. A solution of this strength also exhibits absorption in the violet, and in the blue as far as about λ 4500. In this region, however, the absorption is not strong enough to remove entirely the strong violet line λ 4358, so that the addition of a little potassium bichromate is necessary both to complete the absorption in the violet and to carry the elimination sufficiently far into the blue to remove the green-blue radiation λ 4960. Incidentally this filter cuts out a group of orange-red lines in the neighbourhood of 6200. In the case of the mercury lamp used in the present experiments these radiations were relatively strong only when the lamp had just been switched on when cold. Otherwise they were practically invisible.

Figures 2(*a*) and 2(*b*) exhibit some lack of uniformity in illumination. In these cases, the necessary exposures being short, this uniformity could have been improved by placing a circular stop 2 cm. in diameter immediately in front of the lamp and keeping the condenser *L* perpendicular to the optic axis of the system. The resulting diminution in the amount of light falling on the receiving screen would not have mattered greatly, being corrected for by longer exposure. The much more extended exposures necessary owing to the presence of the filter when the green fringe system alone was being photographed made the arrangement actually adopted preferable for figures (*c*) and (*d*), since thereby little light was lost.

I should like to take the opportunity of expressing my gratitude to Dr L. J. Freeman, in collaboration with whom some earlier experiments on the subject were performed, for his very extensive assistance and interest and for being so kind as to lend me the camera with which these photographs were taken.

DISCUSSION

Dr J. S. Anderson. I think the author is to be congratulated on obtaining such beautiful interferograms with simple apparatus. It would enhance the value of the paper if he would add information regarding the type of mercury vapour lamp used and its current consumption, and also the degree of enlargement of the photographs from the originals. Has the author tried any other type of plate? In work with the Hilger universal lens interferometer I have obtained satisfactory photographs with exposures of about 15 sec., using Ilford rapid chromatic plates and a Wratten and Wainwright mercury green filter, and overrunning the mercury vapour lamp (a quartz one). In photographing Fabry and Perot fringes the conditions are somewhat different and it would not, I presume, be advisable to overrun the lamp. For my work it would be of great advantage to obtain a more sensitive plate, but the grain must be of small size as the original photographs are small and require to be enlarged.

Dr L. C. Martin. It is usually the experience that a quartz mercury vapour lamp is unsuitable for interferometric work on account of the internal pressure developed when the lamp grows hot. I take it that Mr Green used a Cooper-Hewitt lamp for such beautiful photographs as those which he has shown us?

Author's reply. The mercury lamp used was a Cooper-Hewitt lamp, having a glass tube 2 cm. in diameter and carrying a current of 3·3 amp. It was inadvisable to obtain a brighter source of light by overrunning the lamp, in view of the broadening of the spectral lines consequent upon the increase of temperature and pressure. The preliminary set of photographs taken in earlier experiments and shown at the meeting have enlargement factors from the original negatives varying between 1·8 and 2·5, while the positives in Fig. 2 have corresponding values 2·2 in the case of (*a*), (*b*) and (*c*) and 3·5 in the case of (*d*). No systematic test was made with other types of photographic plates.

With regard to the effect of the conditions in the source on the interference pattern produced, perhaps I may mention that on attempting to analyse the mercury radiation from a vertical capillary arc operating at a pressure of about 90 cm. of mercury and carrying 0·3 amp. I found that the fringe system in a Fabry and Perot étalon had vanished almost completely, whereas the Cooper-Hewitt lamp gave satisfactory fringes. This was presumably due to the excessive widening of the spectral lines at the high temperature and pressure of the capillary arc. It is possible that the fringes given by the Cooper-Hewitt lamp might have been further sharpened by reduction of the current strength below 3·3 amp., but any advantage gained would have been largely offset by the necessity for longer exposures.

CHARACTERISTICS OF DISCHARGE TUBES UNDER "FLASHING" CONDITIONS AS DETERMINED BY MEANS OF THE CATHODE RAY OSCILLOGRAPH

By W. A. LEYSHON, Ph.D.

Received November 7, 1929. *Read and discussed January* 24, 1930.

ABSTRACT. An account is given of a method of determining the current-voltage characteristics of discharge tubes under flashing conditions, a cathode ray oscillograph being used. Results are given for neon lamps of the commercial Osglim type. Current/time and voltage/time curves are derived from the oscillograph records. It is suggested that the method might prove useful in investigations of intermittent discharges in various gases under different conditions of pressure and disposition of the electrodes.

§1. INTRODUCTION

Experiments have been carried out during the past two or three years in which a flashing neon tube circuit has been used for the excitation of nerve preparations in order to produce limb movements of natural form. It was thought desirable that some information should be obtained about the nature and duration of the condenser discharges which were utilised for this purpose, and accordingly a number of observations were made with a cathode ray oscillograph. A very brief account of the experimental method and of the results obtained has been given in a recent paper (1).

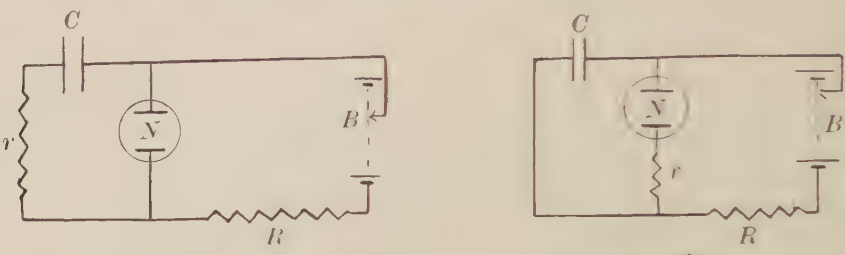
In the course of the experiments results were obtained relating to the characteristics of flashing neon tubes which were not relevant to the original purpose of the investigation, but seemed to be of general interest. It is these results, not previously recorded, which are set out here.

Clarkson (2) in a series of papers has given accounts of experiments on the characteristics of discharge tubes in flashing circuits; the method of experimentation and the circuits which he used differed from those described in the present paper. For the purpose of the physiological experiments it was necessary to include a resistance or an inductance in series with the condenser or the neon lamp in the now well-known flashing circuit first described by M. D. Hart and also by Pearson and Anson. The characteristic curves given here are therefore those for a neon lamp in series with an ohmic resistance, or an inductance, or both, through which a condenser discharges at regular intervals.

Observations have been made on about ten different Osglim lamps, chiefly of the "beehive" pattern; the latter were used in the nerve excitation apparatus, since, with other circuit conditions the same, larger quantities of electricity passed during discharge through this type of lamp than during discharge through other types.

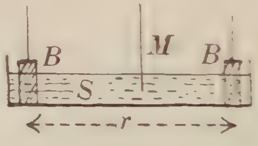
§ 2. DESCRIPTION OF APPARATUS USED

Fig. 1 at (*a*) and (*b*) shows diagrammatically the arrangement of the flashing circuit. The neon lamp *N*, from which the series resistance in the cap has been removed, is in parallel with the condenser *C* which has the non-inductive resistance *r* in series with it, Fig. 1 (*a*), or alternatively the resistance *r* is in series with the neon lamp, Fig. 1 (*b*). The charging circuit consists of a battery of dry cells or high-tension accumulators and a resistance *R* which may vary from 10,000 to 100,000 ohms.



(*a*) (*b*)

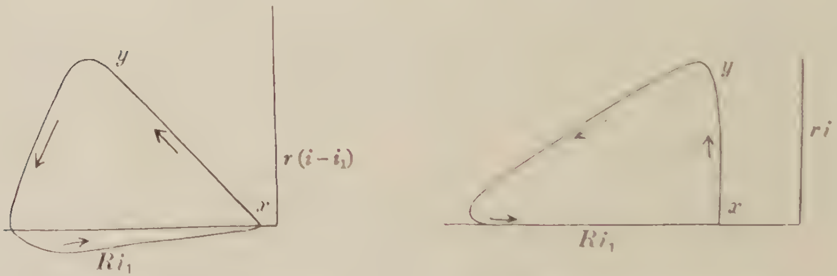
N neon lamp; *R*, *r* non-inductive resistances; *C* condenser; *B* battery.



(*c*)

B, *B* copper blocks; *S* copper sulphate solution; *M* copper wire.

Fig. 1. The flashing circuit.



(*a*) Beehive lamp *N*: $r = 7600$ ohms; $C = 0{\cdot}015\ \mu F$.

(*b*) Lamp *B*5: $r = 1000$ ohms; $C = {\cdot}05\ \mu F$.

Fig. 2. Typical oscillograph figures recorded by tracing.

In the physiological experiments a non-inductive potential divider of the type shown in Fig. 1 (*c*) has been used in the discharge circuit. This consists of two copper blocks, *B*, *B*, in a solution of copper sulphate; the copper wire *M* dipping into the solution acts as the movable point. For the purpose of obtaining the oscillograph records, this arrangement has occasionally been replaced by a copper sulphate liquid resistance of the pattern described by Jordan⁽³⁾.

The potential differences developed across R and r are applied to the two pairs of deflecting plates of a cathode ray oscillograph. The figures seen have been recorded either by tracing or photographically using bromide paper.

It will be observed that in the arrangement of Fig. 1 (*a*) both the charging and the discharge current of the condenser C pass through the resistance r, whereas in the arrangement of Fig. 1 (*b*) the discharge current only passes through r.

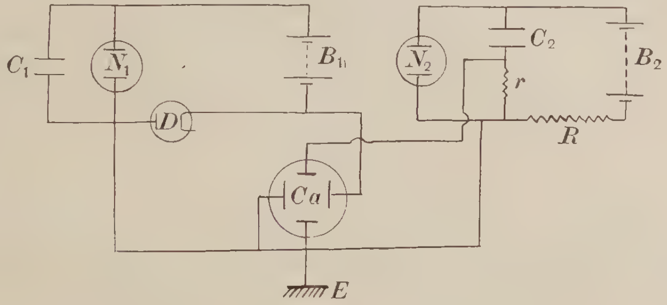


Fig. 3. Apparatus for obtaining discharge-current/time curves.
C_1, C_2 condensers; B_1, B_2 batteries; N_1, N_2 neon lamps;
Ca cathode ray oscillograph; R, r resistances.

Some typical results are shown in Fig. 2. The parts marked XY are rather faint in the actual oscillograph figures owing to the rapidity of travel of the oscillograph spot, and are therefore more difficult to trace accurately than the rest of the figure. Discharge-current/time curves have also been obtained with a neon-tube diode flashing circuit, shown diagrammatically in Fig. 3. The lamp N_1 (usually an I-type lamp) is in series with a diode D (a triode with anode and grid connected) and battery B, the condenser C_1 being in parallel with N_1. Since the charging current to the condenser is by this means kept constant (the filament temperature of the diode being low enough to ensure that the saturation current is taken throughout) the potential difference across the condenser (and therefore that across the diode) changes linearly with time during the charging period. This latter potential difference is applied to one pair of plates of the oscillograph, and affords a time base. To

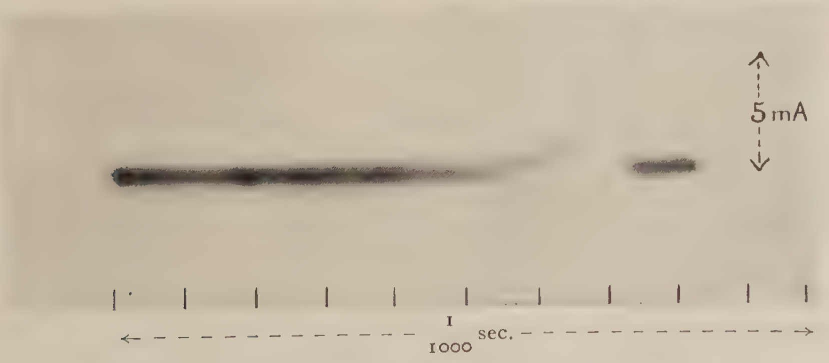


Fig. 3 (*a*). Discharge current (circuit resistance 2000 ohms).

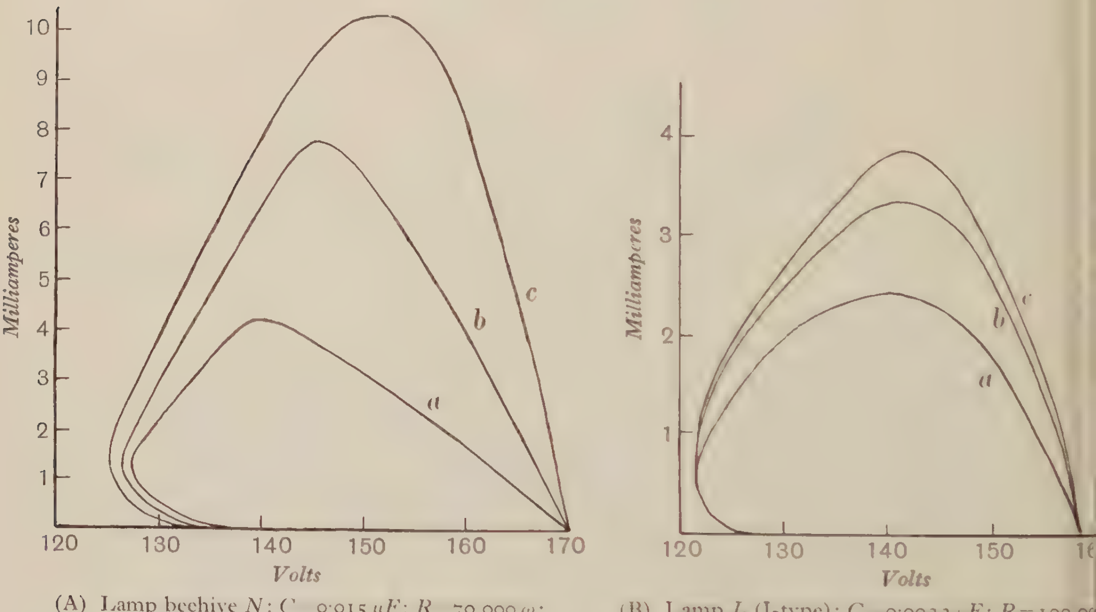
the other pair of plates are connected the two ends of the resistance r in the discharge circuit of the second neon lamp N_2. The lamp N_1 is made to flash at a frequency much higher than that of N_2. In order to keep the oscillograph figure steady, a small coupling is introduced between the two circuits.

The discharge-current time curves, recorded photographically on bromide paper, are found to be of the same shape as those deduced (see § 3) from the records like those of Fig. 2. A typical record is shown in Fig. 3 (*a*).

It may be noted that if the voltage Ri_1 were applied to the oscillograph plates, a record of condenser voltage and time could be obtained, since condenser voltage $= B - Ri_1$. From this it would be possible to construct the characteristic of the lamp when there is no resistance in the discharge circuit.

§ 3. RESULTS

The curves of Fig. 2 show the relation between $r(i - i_1)$ and Ri_1, Fig. 2 (*a*), or ri and Ri_1, Fig. 2 (*b*), where i is the discharge current from and i_1 the charging current to the condenser.

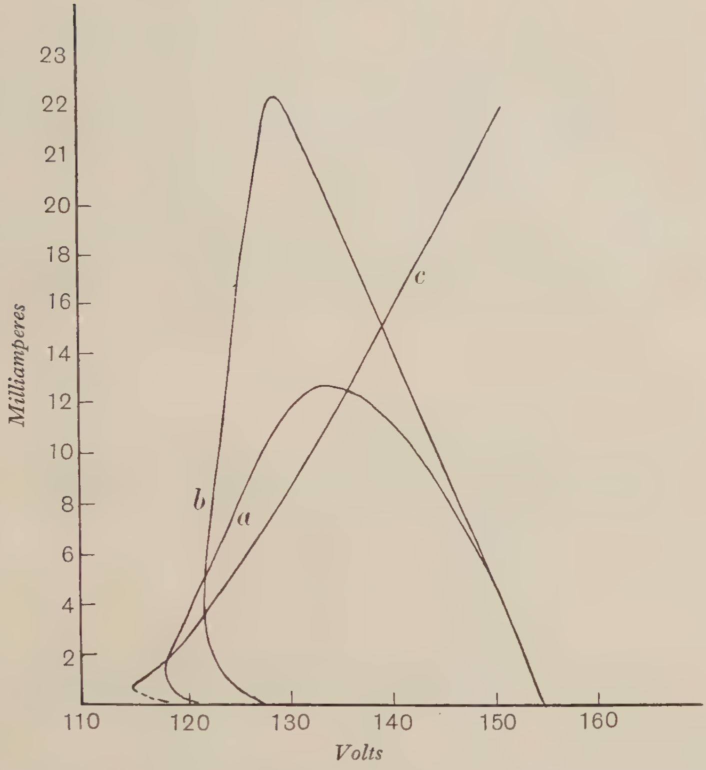


(A) Lamp beehive N: $C = 0{\cdot}015\ \mu F$; $R = 70{,}000\ \omega$; $B = 174{\cdot}4$ volts; (*a*) $r = 7600\ \omega$; (*b*) $r = 2800\ \omega$; (*c*) $r = 880\ \omega$.

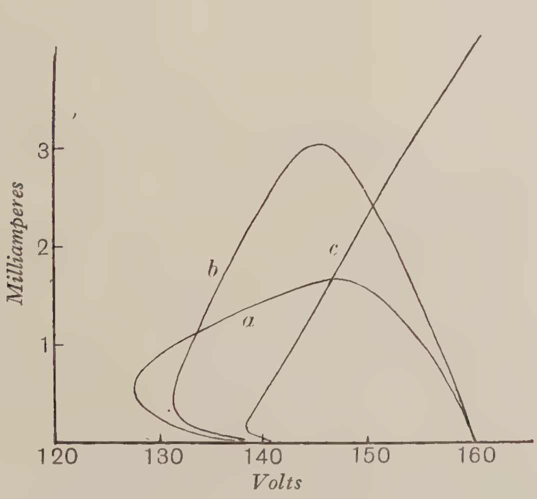
(B) Lamp I_2 (I-type): $C = 0{\cdot}0022\ \mu F$; $R = 100{,}0$ $B = 162$ volts; (*a*) $r = 3000\ \omega$; (*b*) $r = 1850$ (*c*) $r = 1200\ \omega$.

Fig. 4. Characteristics of neon lamp under flashing conditions.

The voltage across the lamp is $(B - Ri_1)$, Fig. 1*a*, or $(B - Ri_1 - ri)$, Fig. 1 (*b*), where B is the battery voltage and i the current through the lamp, so that if B, R, r and the sensitivity of the oscillograph are known the characteristic of the lamp under flashing conditions can readily be derived from the figures. The latter, in fact, for the arrangement of Fig. 1 (*a*), give directly the general form of the characteristic.



(A) Lamp $B5$: $B=156$; $R=100{,}000\,\omega$; $r=1000\,\omega$; (*a*) $C=0{\cdot}003\,\mu F$, $n=614$; (*b*) $C=0{\cdot}05\,\mu F$, $n=72$; (*c*) Static characteristic.

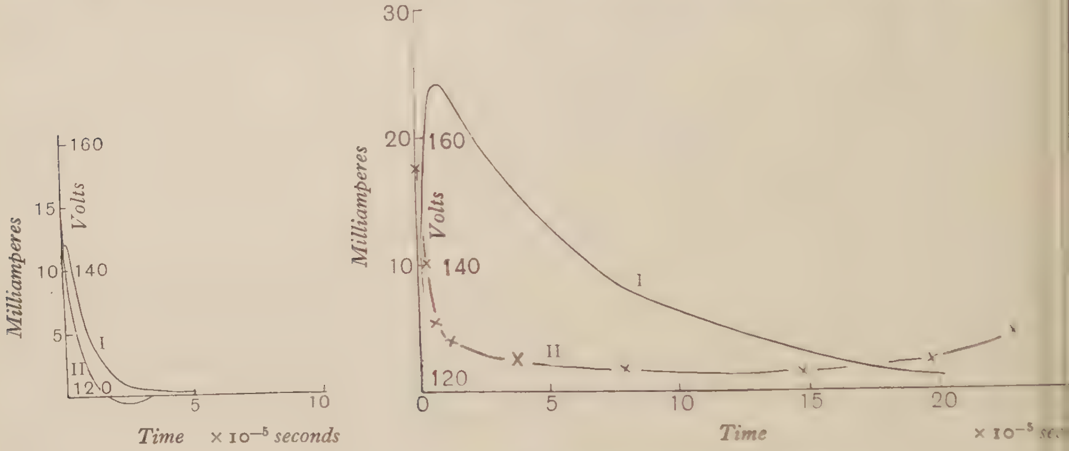


(B) Lamp I_1: $B=162$ volts; $R=100{,}000\,\omega$; $r=3000\,\omega$; (*a*) $C=0{\cdot}0022\,\mu F$, $n=750$; (*b*) $C=0{\cdot}0222\,\mu F$, $n=103$; (*c*) Static characteristic.

Fig. 5. Characteristics of neon lamps: (A) beehive; (B) I-type.

The voltage B was measured under working conditions by means of an electrostatic voltmeter, which was also used in determining the sensitivity of the oscillograph. From the figures and the circuit constants the relation between condenser voltage and resultant current $(i - i_1)$ during the discharge can be found; the general form of the curve is shown in Fig. 2 (*b*). Since $(i - i_1) = C . dV/dt$ the time relations in the circuit can be deduced if C is known.

Fig. 4 shows some typical characteristic curves of a beehive lamp, for which the circuit conditions are stated, for different values of the discharge circuit resistance. Fig. 5 illustrates characteristics of a beehive and an I-type lamp at (A) and (B) respectively. For comparison, the static characteristics of the lamps are also shown. The arrangement used was that of Fig. 1 (*b*). Fig. 6 shows the time relations calculated from the curves of Fig. 5 and the circuit constants.



(*a*) (*b*)

Fig. 6.

(*a*) I. Discharge-current time curve for conditions corresponding to curve *a*, Fig. 5 (A).
II. Lamp-voltage/time curve.

(*b*) I. Discharge-current time curve for conditions corresponding to curve *b*, Fig. 5 (B).
II. Lamp-voltage/time curve ,, ,, ,,

The method of calculation is as follows: The curve showing the relation between condenser voltage V and current $(i - i_1)$ is first drawn. Since $(i - i_1) = C . dV/dt$ we may write $\Delta t = C . \Delta V/[i - i_1]$, where $[i - i_1]$ is the mean value of $(i - i_1)$ in the interval ΔV. Starting with the highest value of condenser voltage, small changes of voltage ΔV are marked out on the $(i - i_1, V)$ graph, and the corresponding times Δt calculated. The sum of these time-intervals gives the time which has elapsed since the initiation of the discharge; thus the relationship between condenser current and time can be determined. From the circuit constants and the figures originally traced corresponding values of condenser current, lamp voltage and discharge current are found, so that the time relations for all the quantities concerned may be traced out. The results may be checked by calculation of the quantity of electricity lost by the condenser during discharge (i) from the area of the $\{(i - i_1), t\}$ curve,

(ii) from $C(V_1 - V_2)$, where V_1 and V_2 are the highest and lowest condenser voltages recorded, respectively.

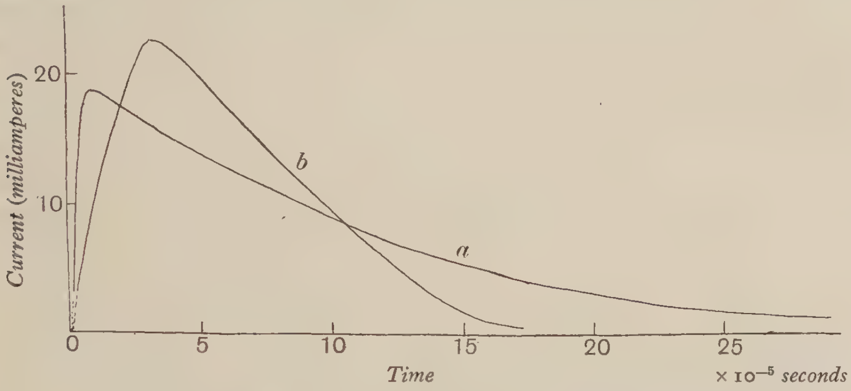


Fig. 7. Typical discharge-current/time graphs for two beehive lamps.
(*a*) Lamp *B*1: $B=174$ volts; $R=60{,}000\ \omega$; $r=1000\ \omega$; $C=0{\cdot}05\ \mu F$
(*b*) Lamp *B*6: $B=192$ volts; $R=70{,}000\ \omega$; $r=1000\ \omega$; $C=0{\cdot}05\ \mu F$

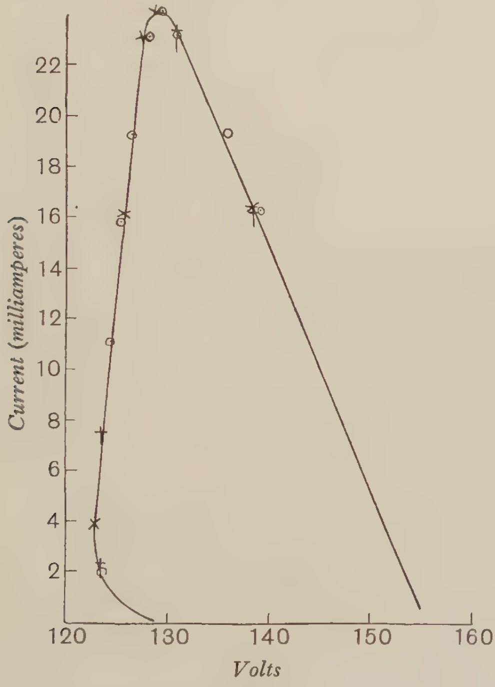


Fig. 8. Characteristic curve for lamp *B*5.

Lamp *B*5
⊙ $R=100{,}000$, $n=72$; $C=0{\cdot}05$.
× $R=80{,}000$, $n=128$; $B=156$.
† $R=50{,}000$, $n=192$; $r=1000$.

Fig. 7 illustrates the difference in the current/time relationships for two beehive lamps, one, *B*6, having a cathode consisting of a more open spiral than the other, *B*1.

Fig. 8 shows that the difference, if any, in the characteristic of a lamp caused by alteration of the charging circuit resistance R is within the limits of experimental error in the tracing of the curves. This result is of importance from the point of view of the physiological application of the circuit, since it shows that the rate of flashing can be altered without materially affecting the quantity of electricity passing during each single condenser discharge.

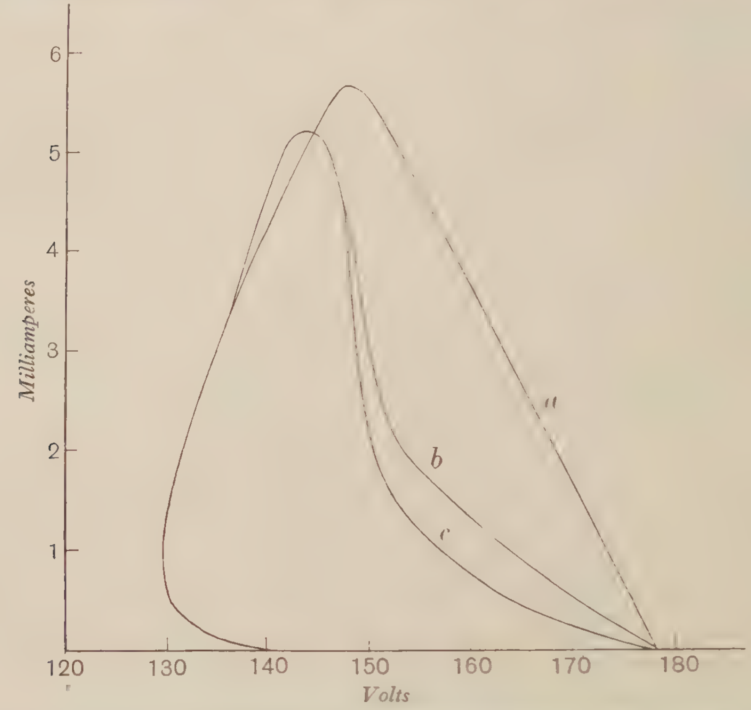


Fig. 9. Effect of inductance in discharge circuit on characteristic curve.

Lamp beehive N: $C = 0{\cdot}01\ \mu F$; $B = 178$ volts; $r = 5240\ \omega$; (*a*) No inductance in discharge circuit; (*b*) Two No. 400 Igranic coils in discharge circuit; $R = 70{,}000\ \omega$; (*c*) Two No. 400 with iron core in discharge circuit; $R = 60{,}000\ \omega$.

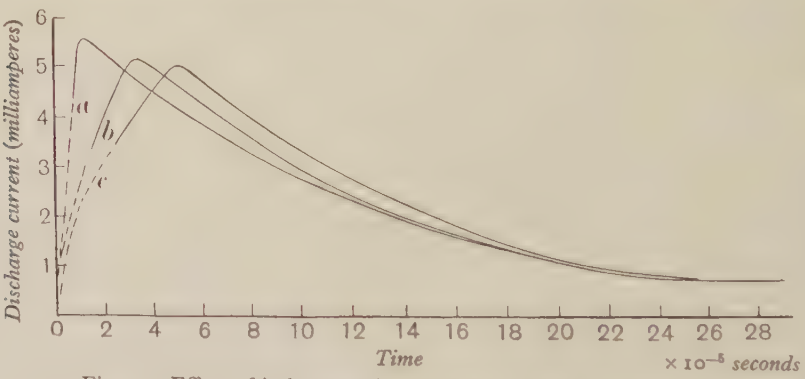


Fig. 10. Effect of inductance in discharge circuit on (i, t) curve.

It is clear from the results given, and from a large number of similar observations which have been made, that (1) The range of alteration of voltage across the lamp is decreased, and the shape of the characteristic is altered, when the condenser capacity is raised (see Fig. 5 (A) and 5 (B)). This range is not affected to any great extent by changes in battery voltage or in series charging resistance—at least within the limits of these experiments, and provided that the applied voltage be greater than the normal striking voltage. (2) With large values of condenser capacity the voltage across the lamp remains almost constant during the greater part of the discharge, after the initial rapid drop. (3) The presence of inductance in the charging circuit is found experimentally to lower considerably the critical resistance for flashing. This has been previously noted (4).

The effect on the lamp characteristic of inductance in the discharge circuit is shown in Figs. 9 and 10. With a large inductance the current may rise very sharply while a small drop in voltage occurs: i.e. the negative slope resistance may be very great.

§4. CONCLUSION

The method given for obtaining characteristic curves of neon lamps involves the use of no more apparatus than the flashing circuit itself and the cathode ray oscillograph. So far it has been applied only to the commercial type of lamp, but it is clear that it could readily be applied to the study of intermittent discharges in pure neon and other gases, under different conditions of pressure and disposition of the electrodes. The use of the cathode ray oscillograph avoids the errors introduced by other types of recorders, such as that used by Anthouard (5) in his work on intermittent discharges through air.

§5. ACKNOWLEDGMENTS

The experiments described have been carried out in the Physics Laboratory of the London School of Medicine for Women. I should like to express my thanks to the Council of the School for grants from the Waller Memorial Research Fund for the purchase of apparatus.

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DISCUSSION

Dr D. W. Dye. I wish to congratulate the author on the execution of a very useful piece of work which has given definite results, while her methods have shown promise of usefulness in allied investigations. I have not had much experience of discharges of this nature in neon, but some years ago I tried such tubes giving intermittent discharges to see if they could be used as sources of discontinuous oscillations from which high-order harmonics could be selected to serve as radio frequency standards. I was compelled to discard the type of tubes then available because it was found that within one cycle the discharge presented discontinuities which were irregular, and so clouded over the precise selection of a harmonic. It appears, however, from the author's diagrams that the cycle is smooth and regular so that it is difficult to see how the results I obtained could agree with this.

Again recently I tried the use of intermittent discharges from neon tubes to serve as a stroboscopic source of illumination of interference fringes produced between two surfaces, one of which was vibrating rapidly. In this case also no sharpness of definition was possible, but whether owing to luminous persistence of the discharge or to irregular discontinuities I could not say.

Dr A. B. Wood. In her introductory remarks Dr Leyshon mentioned that the object of the experiments was to determine the wave-form of the oscillations in the neon lamp circuit. The voltage/current characteristic, whilst serving this purpose indirectly, does not appear to me to afford the most satisfactory method. Of course if a non-photographic oscillograph, such as the Western Electric type, is used, this method is the only one possible. Some years ago (1924), however, I obtained a series of cathode-ray oscillograph records of the oscillations of an Osglim (neon-filled) lamp, using the circuit shown in Fig. 11. This is the Anson-Pearson flashing circuit in which two condensers C_1, C_2 are used in series to form a potential divider for oscillograph purposes. With this arrangement voltage time curves of the commencement of oscillations in the circuit were recorded in a single traverse of the cathode spot across a photographic plate in the vacuum chamber of the oscillograph. Such records are shown in Fig. 12. The frequency of oscillation of the circuit in these cases varied from 1000 to 3000 ~. The following points are of interest: (1) The voltage across the lamp at first rises (during the condenser charge) to a high striking potential, at which point it suddenly falls, recovers, and continues to oscillate. The mean voltage of the sustained oscillation is, as would be expected, considerably lower than the striking voltage. (2) The higher the frequency the smaller the voltage-amplitude of the oscillations. Ultimately it was found that the lamp would not oscillate at frequencies of the order of 100,000 ~ in the circuit shown in Fig. 1. (3) As Dr Dye has already suggested, the initial rise of voltage, before the lamp begins to glow, is not always regular. Some of the records show distinct irregularities in the rising voltage. (4) The wave form is typical of B. van der Pol's "relaxation" oscillations—a succession of slow rises and sudden falls of voltage. I cannot understand why the previous speaker failed to observe the presence of harmonics in such a wave-form as that indicated in the records.

The President. The use of the cathode ray oscillograph in examining intermittent discharges might help in the elucidation of some of the problems of conduction of electricity through gases left unsolved by the more orthodox modes of experiment. I understand that Dr Leyshon is proceeding with these investigations but employing simpler electrode systems, more amenable to calculation than those in the commercial forms of neon lamp.

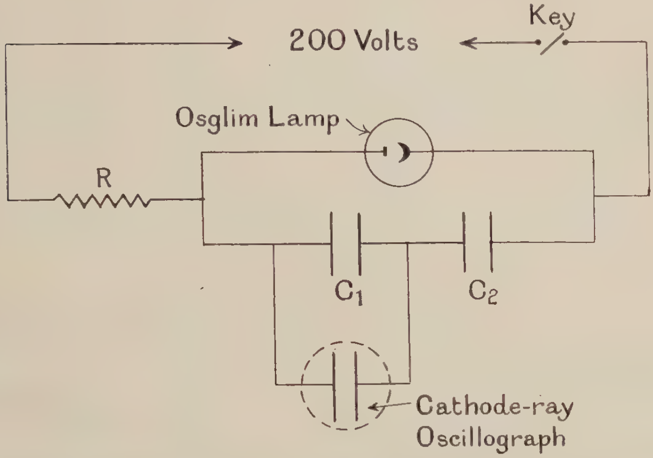

Fig. 11. Anson-Pearson flashing circuit—oscillograph recording voltage/time characteristic.

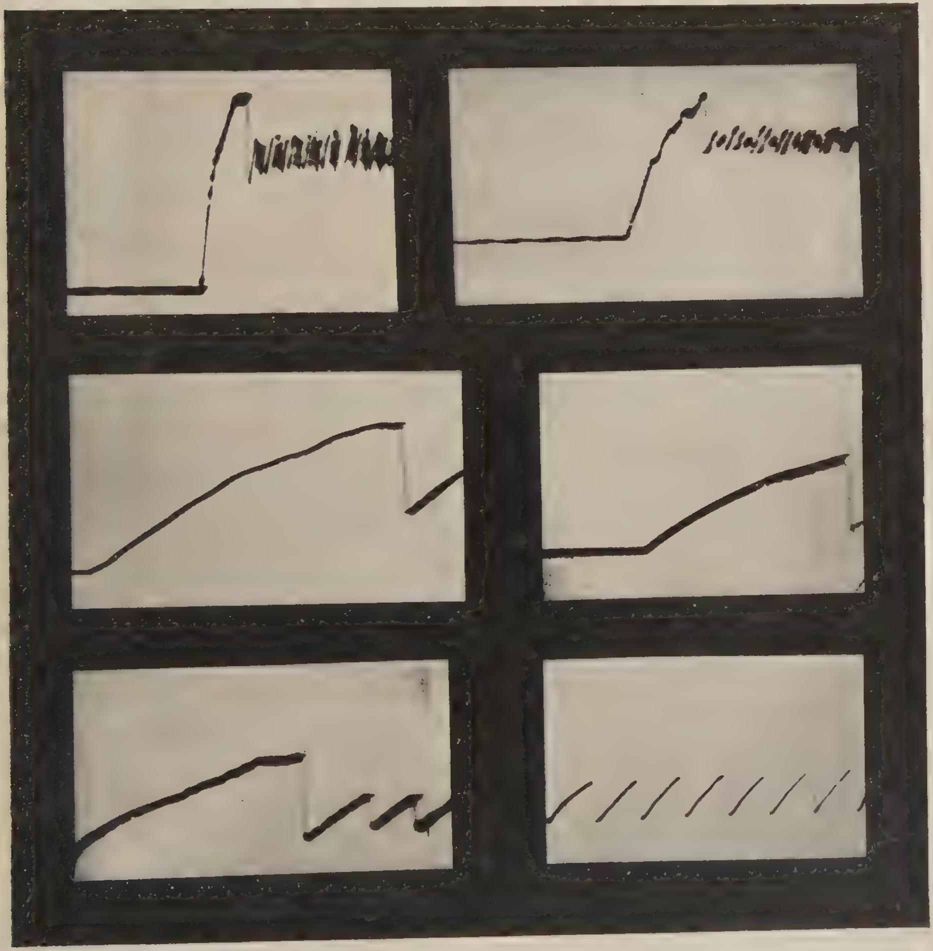

Fig. 12.

Mr L. G. Carpenter. In view of the growing importance of the cathode-ray oscillograph as a means of investigating transient phenomena, it would enhance the value of the author's very interesting paper if she would add some details as to the characteristics of the cathode-ray oscillograph she used. Any information as to the best material for the fluorescent screen in visual observations of the spot made by a rapidly moving cathode ray stream would be of especial interest. With regard to the bromide paper, where precisely was it placed, and was it specially treated in any way?

Mr N. L. Harris. Dr Leyshon's interesting paper sets out a useful method of obtaining dynamic discharge tube characteristics. The striking and extinguishing potentials of the lamps as given by Figs. 4 and 5 are some 15 or 20 volts lower than it is usual to find amongst a normal batch of Osglim lamps. In particular, the value of 115 volts for the extinguishing potential of lamp B_5 in Fig. 5 (A) is very unusual: possibly the readings of the electrostatic voltmeter are subject to a correction of this order. The constant voltage across the lamp during discharge under certain conditions is probably due to the fact that the cathode fall of potential is "normal," with the cathode incompletely covered with glow. It is well known that in these circumstances the tube voltage is almost independent of the discharge current. In connection with Dr Leyshon's recent demonstration of periodic movements of the cathode glow, and Dr Dye's remarks upon the apparently irregular growth of the flash, it might be advisable to emphasise the important part played by surface films on the electrodes. The striking of the discharge is intimately associated with the electronic work function of the cathode surface, and this in turn with any adsorbed films of monomolecular nature at this surface. A small patch of such a film of an electropositive element is sufficient to initiate a local discharge at a lower potential than that required for the clean electrode surface, and small traces of an electronegative element have the reverse action. It is wellnigh impossible to maintain a discharge tube free from the traces of impurity sufficient to cause such effects. The presence of such films, which change constantly under the influence of the discharge, plays an important part in bringing about many of the phenomena described. The existence of these films is demonstrated very simply by the increase of striking potential noticed after a discharge tube has stood unused for some hours. An adsorbed film of gaseous impurities will then have formed on the cathode, and the striking voltage will be raised. The passage of the discharge cleans the surface, in part at least, and the striking potential falls.

Author's reply. I should like to thank the various speakers for their encouraging comments on and criticisms of the paper.

The discontinuities in individual discharges mentioned by Dr Dye were not observed in my experiments -no doubt because each recorded figure represented an average of some thousands of discharges. The distinctness of the figures seen showed, however, that the variation of the greater number of individual discharges from the average could not have been very marked—at least after the initial rapid rise of current. It is possible that the resistance in the discharge circuit may have had a stabilising effect.

The original object of the investigation was to obtain some idea of the nature and duration of the condenser discharges through the neon lamp, these repeated discharges having proved to be a suitable form of stimulus for excitation of nerve preparations.

In Dr Wood's interesting records the voltage changes across the condenser during discharge are shown by the faint lines, which illustrate the difficulty in recording discharges of such brief duration by a method involving a single traverse of the oscillograph spot across a photographic plate.

The explanation for the change in striking voltage which is shown so clearly in Dr Wood's records has been given by Mr Harris in his contribution to the discussion.

The electrostatic voltmeter I used gives readings which correspond closely with those of a Weston instrument and of a Cambridge Instrument Company's voltmeter. The lamp $B5$ for which records are given in Fig. 5 (A) certainly has a lower minimum voltage than any other lamp which was examined. I have lately redetermined the minimum voltage for this particular lamp, and have confirmed previous observations. The shapes of the characteristics of this lamp are, however, like those of other beehive lamps.

In reply to Mr Carpenter, the cathode ray oscillograph used was of the standard Western Electric type (No. 224 A). Its sensitivity was about 1 mm. movement of the oscillograph spot for 1 volt potential-difference between the deflecting plates. The bromide (gas light) paper used was not specially treated. It was held down on the top of the oscillograph tube for about half a minute. This method of obtaining photographic records with bromide paper was described by Dr Dye in a paper read before the Physical Society*.

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* *Proc. Phys. Soc.* **37**, 158 (1925).

A METHOD OF CALCULATING THE NUMERICAL EQUATION OF STATE FOR HELIUM BELOW 6° ABSOLUTE, AND OF ESTIMATING THE RELATIVE IMPORTANCE OF GAS DEGENERACY AND INTERATOMIC FORCES

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ABSTRACT. Gas "degeneracy" (the effect of non-Maxwellian distribution of molecular velocities) is distinguished from gas "imperfection" (the effect of intermolecular forces), and it is shown that specific heat measurements cannot be used to verify theories of the former, unless account is taken of the very rapid increase of the latter at the lowest temperatures and highest pressures. Accordingly a thermodynamic method is used to calculate the sum of degeneracy and imperfection at 4° and 5° Abs. for helium. Theories of degeneracy can then be roughly tested by the extent to which they allow a residual value of the imperfection to be calculated, and determine a consistent extension of the course taken by isothermals down to 15°. The method is applied to Fermi's expressions for degeneracy, and also to Berthelot's equation of state. One of the equations alone satisfies the required condition; it is shown that on this basis degeneracy would comprise 15 per cent. of the total departure of helium from the ideal gas laws at 4° and 5°, the remainder being due to true imperfection or intermolecular forces.

§ 1. STATEMENT OF THE PROBLEM

LET the ideal gas be defined, for the present purposes, by its obedience to the equation

$$p = n\kappa T \qquad \text{......(1)},$$

where p is its pressure in dynes cm.^{2}, T its absolute temperature, n the number of molecules per c.c., and κ is $1{\cdot}372 \times 10^{-16}$.

Any actual gas departs from this law owing to the finite size of its molecules and the forces of attraction and repulsion between them, and we follow normal usage in referring to this departure as the "imperfection" of the gas. But for many years in thermodynamics (Nernst's heat theorem) and more recently in kinetic theory (the inclusion of translatory motion under quantum restrictions) it has seemed probable that even a gas which had no imperfection in the above sense would nevertheless still show departure from law (1) at high pressure and low temperature. In statistical mechanics this accompanies a departure from Maxwellian distribution of molecular velocities, and has been called the "degradation" or "degeneracy" of a gas.

To include both the imperfection and the possible degeneracy of a gas, its true equation of state would have to take the form

$$p = n\kappa T\{1 + F(T, p) + f(T, p)\} \qquad \text{......(2)},$$

where $F(T, p)$ is a function of temperature and pressure representing the effect of molecular dimensions and forces, and $f(T, p)$ represents the effect of departure from Maxwellian statistics.

For brevity we shall refer to these two functions as A and B respectively.

The very large number of empirical equations of state which have from time to time been proposed, beginning with that of Van der Waals, deal exclusively with A, and there is no doubt that B is only likely to become important under the most extreme conditions of pressure and temperature realisable in practice. Attempts to verify the predictions of degeneracy under those conditions made by thermodynamic and kinetic theory have mainly been based on the following argument. Even the most complex equation of state must satisfy the general relations

$$E = p/\tfrac{2}{3}n,$$
$$C_v = (dE/dT)_v,$$

where E is the energy of the n molecules and C_v the specific heat at constant volume. Hence degeneracy must affect C_v, and anomalies discovered in calorimetry may give information about B if the substance can be made to remain in the gaseous phase at low enough temperature. Hydrogen and helium obviously present the most favourable cases, because of their comparatively slight imperfection and consequently low boiling points. At temperatures so low that the contribution of rotational and vibrational degrees of freedom to the specific heat have vanished, both these gases should, in the ideal state, obey the law

$$C_v = \tfrac{3}{2}\kappa.$$

The measurements of Eucken⁽¹⁾, and more recently of Meissner⁽²⁾, reveal a distinct fall below this value for both gases at very low temperatures, and hence these experiments have often been quoted as experimental verification of degeneracy.

For hydrogen this interpretation is complicated by the recently found difference between the rotational specific heats of the ortho- and para-molecular varieties, at temperatures just above that at which those quantities vanish⁽³⁾. However, it is true that any survival or anomalous excitation of rotational degrees of freedom could only increase C_v, and so cannot be responsible for the facts; further, all theories of degeneracy do predict a tendency in the observed direction by their requirement that

$$C_v \to 0 \quad \text{as} \quad T \to 0.$$

Nevertheless, for any such experiments to afford a convincing test of theoretical forms of B, or even a decision as to whether B is large or small under the most extreme practical conditions, the course taken by the function A would have also to be known in this region.

Such a requirement is not easy to fulfil; we shall indicate below some inconsistences produced by extrapolating, to temperatures at which degeneracy is likely to be important, empirical equations of state known to represent A satisfactorily under less extreme conditions. We are left with the suspicion that effects attributed to B may be due to an unknown increase in A, and any independent clue as to the relative importance of imperfection and degeneracy becomes worthy of enquiry. Hence in the present paper the problem of the possible place of degeneracy in experiment is investigated from a different standpoint, as follows.

§ 2. METHOD OF THE PRESENT PAPER

A method is devised whereby the vapour pressure and latent heat data of the Leiden experimenters are made to yield values of $(A + B)$ through the latent heat equation of thermodynamics. Now direct measurement of helium isothermals has been carried down to 15° Abs. by the Berlin Reichsanstalt. From their data we plot a curve of the departure of the gas from the ideal law between 90° and 15°. This represents the course of $(A - B)$, but it is very probable that, even at 15° and the given pressure, B is small compared with A, and that the curve substantially represents the course of A. We show that the further points secured by our method at 4° and 5° lie on a smooth continuation of this graph of $(A + B)$. But whereas B was probably negligible at 15°, it is not necessarily so at the lower temperatures. Accordingly it becomes of interest to substract from our values of $(A + B)$ any given theoretical value of B; it is then possible to see at once whether the residual value of A is or is not still consistent with the course entered upon by $(A + B)$ at the higher temperatures for which B was negligible. We can in this way make a rough test of any theory of degeneracy, by requiring that the value of A which it extracts from our $(A + B)$ must not lie in such a part of the diagram as to bring obvious discontinuity into the course of that function.

This indirect test is capable of eliminating impossible theories rather than of distinguishing between those which satisfy the conditions, since one extrapolation can be shown to be wrong when another cannot be proved to be right. It does not, however, involve the assumption that an equation of state has equal validity on both sides of the critical temperature, and thus it may have some advantage over the only alternative method by which theories of degeneracy can be compared with experiment.

The method is applied to Fermi's two expressions for degeneracy. A similar test is made of the validity of Berthelot's equation of state in this region.

§ 3. TOTAL DEPARTURE FROM GAS LAWS AT 4·29° ABS. AND 5·20° ABS., DERIVED BY INDIRECT METHOD

If equation (2) be written in the form

$$p = n\kappa T(1 + A + B),$$

where A and B are the imperfection and degeneracy functions of T and p, the total $(A + B)$ can be calculated if pressure and density and atomic mass are known, since we have

$$n = \rho/m.$$

Consider the latent heat equation

$$L = (1/\rho_V - 1/\rho_L)\, T\, (dp/dT),$$

where L is the sum of external and internal latent heats, ρ_V and ρ_L are densities in the vapour and liquid phases respectively, and p is the saturated vapour pressure at temperature T. Kammerlingh Onnes⁽⁴⁾ determined the density of liquid helium at 4·29° Abs., and the latent heat of evaporation at the same temperature⁽⁵⁾. From the vapour pressure experiments of Onnes and Weber⁽⁶⁾ we construct a graph giving us the pressure and pressure gradient at that temperature. From all these sources we obtain the following:

$$T = 4{\cdot}29^\circ,$$
$$L = 22 \text{ cal./gm. mol.}$$
$$= 2{\cdot}30 \times 10^8 \text{ erg./gm.},$$
$$\rho_L = 0{\cdot}122 \text{ gm./c.c.}$$
$$dp/dT = 9{\cdot}37 \times 10^5 \text{ dyne/cm.}^2 \text{ per degree},$$

whence we calculate from the latent heat equation

$$\rho_V = 0{\cdot}0152 \text{ gm./cc.},$$

a result which is not inconsistent with the statement from the other cryogenic laboratory⁽⁷⁾ that at a slightly higher temperature the density of the vapour is about one-tenth of that of the liquid helium.

For the mass of the helium atom we have

$$m = 6{\cdot}56 \times 10^{-24} \text{ gm.}$$

For Boltzmann's constant

$$\kappa = 1{\cdot}372 \times 10^{-16}.$$

Hence at 4·29°

$$n\kappa T = 1{\cdot}361 \times 10^6.$$

From the curve plotted with the data of Onnes and Weber, at this temperature,

$$p = 810 \text{ mm. Hg.}$$
$$= 1{\cdot}082 \times 10^6 \text{ dynes/cm.}^2.$$
$$\therefore A + B = -\,0{\cdot}204.$$

The total departure from the ideal gas law can again be calculated at 5·20° Abs. This is the critical temperature at which L vanishes. But, using the critical coefficient for helium, we have

$$\kappa T_c/p_c V_c = 3{\cdot}13,$$
$$p_c = 1718 \text{ mm. of mercury}$$
$$= 2{\cdot}29 \times 10^6 \text{ dynes/cm.}^2.$$
$$\therefore V_c = 99{\cdot}4 \times 10^{-24} \text{ cc.}$$

This is the critical volume for a single atom, since we have used the above "atomic" value for κ. We accordingly combine it with the above value of the atomic mass and obtain

$$\rho = 0{\cdot}066 \text{ gm./c.c.}$$

Hence at 5·20°

$$n\kappa T = 7{\cdot}20 \times 10^6.$$

$$\therefore\ A + B = -0{\cdot}682.$$

§4. TOTAL DEPARTURE FROM GAS LAWS AT HIGHER TEMPERATURES

These indirect determinations of the total departure from the ideal state at 4·29° and 5·20° must be compared with direct determinations at the lowest temperature available. In 1926 Holborn and Otto(5) at the Reichsanstalt investigated isothermals for helium down to 15° Abs.; we calculate from their data the course taken by $(A + B)$ at the two pressures we have been considering, obtaining two curves over the range 90° to 15°. Beyond this range no results are obtainable except by indirect methods. The positions of points representing our values at 4° and 5° are therefore of some interest; we enquire later whether their fit on to the curves is improved or spoilt by assuming any given theory of degeneracy.

Below 60° Abs. the change in the helium isothermals is so rapid that considerable care has to be taken in choosing any way, graphic or algebraic, of enabling their course to be shown without ambiguity. The usually adopted method of plotting a first-order correction term to Boyle's law is not adequate to this case. We take (Table 1) the data of Holborn and Otto from their empirical expression,

$$pv = A + Bp + Cp^2 + Dp^3 + Ep^4.$$

Table 1. Coefficients in Holborn and Otto's equation.

Absolute temperature	A	$B \times 10^3$	$C \times 10^6$	$D \times 10^6$	$E \times 10^9$
90·1	0·32992	0·62286	0·735	—	—
65·1	0·23847	0·55080	2·384	−0·0141	—
20·3	0·07460	−0·16424	18·529	−0·1105	—
15·1	0·05558	−0·79740	54·370	−0·7513	3·796

After unsuccessful attempts to make an unambiguous extrapolation of each coefficient separately, logarithmically and otherwise, it was found more definitive to calculate the integrated value of pv and subtract it from the normal value of the ideal gas. We plot the differences $(pv - RT)$ as a function of temperature for the two conditions with which we have been concerned, namely, 810 mm. and 1718 mm. The result, transferred into the units of Holborn and Otto, is shown in Fig. 1, the unit of pressure being one metre of mercury, while the unit of volume is the volume of one gram of gas at that pressure and 0° C.

From the numerical equation of state which we have derived at the lower temperatures, we determine, transferring our values to the units of Holborn and Otto, comparable expressions for $(pv - RT)$ at 4·29° and 5·20° and the pressures 810 mm. and 1718 mm.

It is seen that these points (+ in Fig. 1) fit a not unreasonable extension of the curves beyond the lowest temperatures at which direct data are available. Since the curvature of both graphs is increasing rapidly at 15°, more than one set of values at 4° and 5° might be regarded as agreeing with their extrapolation. Here again logarithmic plotting was not found to remove possible ambiguity, and the rigorous identification of any extrapolation would require an empirical equation involving higher order terms to which no clue is available at less extreme temperatures. Hence arguments from these curves are only valid in particular cases where decision has to be made between very divergent alternatives.

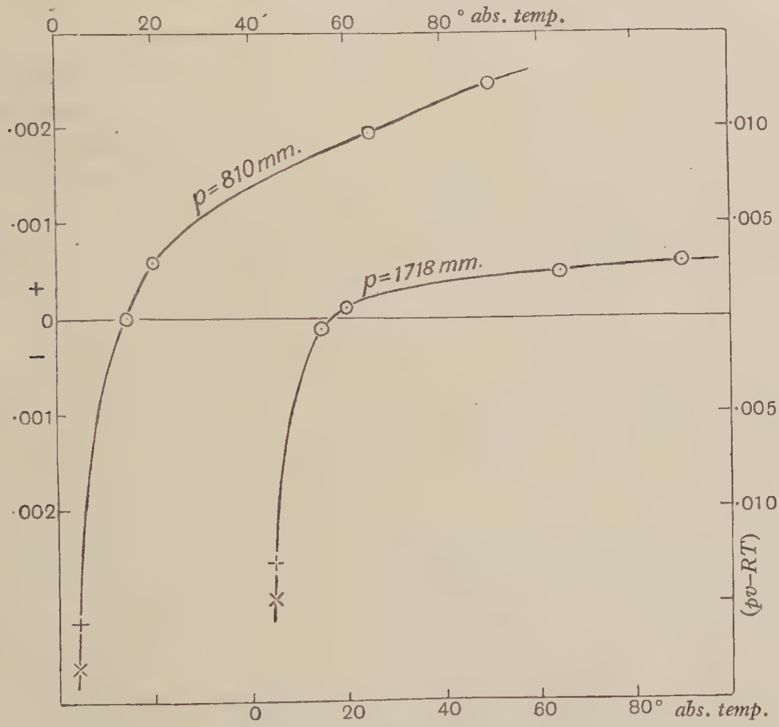

Fig. 1. Gas imperfection, degeneracy and temperature.

So long as this restriction is regarded, we can say that, if a single curve from 4° to 90° represents $(A + B)$ and over the upper half of it B is negligible, a broad limit can at once be set to the allowable degeneracy. Either B at the lowest temperature must be small enough not to alter our extension of $(A + B)$ appreciably, or else if B is large A must be supposed to have increased, with opposite sign, enough to prevent their algebraic sum from departing noticeably from the curve. In the particular case discussed below, the numerical values are such as to permit decision between the alternatives.

§ 5. THEORETICAL VALUES OF DEGENERACY

We turn to theoretical expressions for the function B, of which the basis has been twofold.

(*a*) Degeneracy arises from the application of Nernst's heat theorem to non-condensed systems, and is there inseparable from Nernst's view of the existence of zero-point energy. Treated thermodynamically, the most complete exploration of its consequences has been that of Bennewitz, who, however, did not succeed in deriving in quantitative form a function which we could treat as B for any finite temperature and pressure.

(*b*) Independently of Nernst's heat theorem, Einstein, Bose, Dirac, Fermi and others have investigated degeneracy as a necessary consequence of statistical theorems on the non-Maxwellian distribution of molecular velocities. Confidence in these theorems is mainly due to their success, in the hands of Sommerfeld and others, in explaining otherwise intractable phenomena in other branches of physics. We select as a case for treatment by the present method two approximations derived by Fermi$^{(9)}$ for "strong" and "weak" degeneracy, and expected to describe phenomena under the most extreme and less extreme conditions respectively. The border-line between the regions to which they could apply, and indeed their validity anywhere, has not been otherwise tested.

Strong degeneracy: Fermi predicted the following relation between p and n, κ, T,

$$p = 0\cdot077\, h^2 n^{5/3}/m + 7\cdot6\, m\, n^{1/3} \kappa^2 T^2/h^2,$$

in which I have reduced to numerical constants certain elements in his expression which are independent of physical data. The only symbol which has not previously been defined is h, Planck's constant, equal to $6\cdot55 \times 10^{-27}$.

Weak degeneracy:

$$p = n\kappa T\,\{1 + h^3 n/16\,(\pi m \kappa T)^{3/2}\}.$$

We insert in these equations numerical values appropriate to the cases we have considered, namely,

$$T = 4\cdot29^\circ \qquad \rho = 0\cdot0152,$$

$$T = 5\cdot20^\circ \qquad \rho = 0\cdot066,$$

and compute the degeneracy function for an otherwise perfect gas in the form

$$p = n\kappa T\,(1 + B).$$

If weak degeneracy occurs,

$$B = +\,0\cdot031 \quad \text{at } 4\cdot29^\circ$$

$$= +\,0\cdot10 \quad \text{at } 5\cdot20^\circ.$$

If strong degeneracy occurs,

$$B = +\,3\cdot48 \quad \text{at } 4\cdot29^\circ$$

$$= +\,1\cdot30 \quad \text{at } 5\cdot20^\circ.$$

§6. INFERRED VALUES OF "IMPERFECTION"

Subtracting these theoretical values of B from the values of $(A+B)$ determined by our indirect method, we find that:

weak degeneracy being assumed,

$$\begin{aligned} A &= -0{\cdot}235 \quad \text{at } 4{\cdot}29^\circ \\ &= -0{\cdot}78 \quad \text{at } 5{\cdot}20^\circ; \end{aligned}$$

strong degeneracy being assumed,

$$\begin{aligned} A &= -3{\cdot}68 \quad \text{at } 4{\cdot}29^\circ \\ &= -1{\cdot}98 \quad \text{at } 5{\cdot}20^\circ. \end{aligned}$$

Applying the method of test outlined previously, we see that strong degeneracy, if existent in this case, would cause the imperfection to be 18 times and 3 times the *total* departure from the ideal law at 4° and 5° respectively. Now the rates of increase of the curves drawn from the data of Holborn and Otto, regarded as substantially graphs of A, are such that our values might still appear as extrapolations thereof, even after adjustments by, say, 50 per cent. or even 100 per cent. But adjustment by factors of 3 and 18 would introduce into them a character having no connection at all with their courses at 15°. Further, it would involve the arbitrary assumption that our values of $(A+B)$ are due to a remarkable coincidence, the degree of neutralisation of $+3{\cdot}48$ by $-3{\cdot}68$ having the same proportional effect on the one curve as the degree of neutralisation of $+1{\cdot}30$ by $-1{\cdot}98$ had on the other. This would be a very forced explanation, compared with the simple one that B is small compared with A. Again, if strong degeneracy held at 4° and 5°, the above figures show that it would make the value of A at 810 mm. much greater than at 1718 mm. Reference to the curves, which are on suitably different scales because of the greater order of magnitude of A at 1718 mm. than at 810 mm., shows the extreme unlikelihood of this being so. The higher-pressure curve increases much faster than the lower, and we have no right to demand that they should converge again, since the mechanism of imperfection must be the same in both.

On the other hand, the expression for weak degeneracy introduces a correction to $(A+B)$ which is more convincing, in that it enables the Holborn and Otto curves to be extended in the same relationship to one another which they had already shown. The rate of increase of B relative to that of A is then similar on each curve. In fact at each pressure the degeneracy amounts to about 15 per cent. of the total departure from the behaviour of an ideal gas. The degeneracy correction to the pressure is about 3 per cent. and 10 per cent. at $4{\cdot}29^\circ$ and $5{\cdot}20^\circ$, the pressures being rather over one and two atmospheres, while for the imperfection correction the corresponding figures are of the order of 20 per cent. and 80 per cent. and of opposite sign. In Fig. 1 the points × represent A, assuming weak degeneracy.

Owing to this opposite sign, if conditions are reached at which the strong approximation becomes valid, we may expect a cancellation of imperfection by degeneracy, and a second "Boyle" point, at which again the behaviour approximates to that of the ideal gas through the condition

$$A = -B.$$

§7. EXTRAPOLATION FROM BERTHELOT'S EQUATION

It has been stated above that in this type of problem it is unsafe to assume that any equation of state is valid when approaching the absolute zero, however well it may fit the facts in another range. It is of interest to see what results would be obtained if this precaution were neglected and A calculated by extrapolating some equation known to be a good approximation at temperatures high compared with the critical temperature. Berthelot's equation is the most successful of these; it may be stated in the form,

$$pv = RT\left\{1 + \frac{9}{128}\frac{T_c}{T}\frac{p}{p_c}\left(1 - \frac{6T_c^2}{T^2}\right)\right\}.$$

We calculate the values of A corresponding to those already deduced, inserting the appropriate values of T_c and p_c and taking $T = 4{\cdot}29$ and $5{\cdot}20$ and $p = 810$ mm. and 1718 mm., and obtain

$$A = -0{\cdot}314 \quad \text{at } 4{\cdot}29^\circ$$
$$= -0{\cdot}351 \quad \text{at } 5{\cdot}20^\circ.$$

As might be expected for an equation designed for less extreme circumstances, this predicts a less rapid increase of departure from the ideal law than that found by the indirect method, whether degeneracy is assumed or neglected.

§8. CONCLUSION

Since an unknown increase of gas imperfection at very low temperatures might be responsible for calorimetric effects attributed to gas degeneracy, the need arises for indirect methods of checking theories of degeneracy and of estimating the numerical importance which any of them would bear, relative to imperfection, in the most general equation of state. The method outlined in the preceding sections allows the total departure from ideal gas laws to be computed from vapour pressure and latent heat data for helium at 4° and 5° Abs.

In comparison, Berthelot's equation of state fails to give a sufficiently rapid increase of imperfection. Fermi's "strong" approximation to the degeneracy function also fails to justify the assumption that it can occur at these temperatures and pressures, since it would require too rapid an increase in the imperfection function, and would require the more rapid increase to take place at the lower pressure. On the other hand, Fermi's "weak" approximation provides a degeneracy consistent with curves constructed from direct data above 15° and indirect data below 15°.

The proof that Fermi's weak degeneracy is not inconsistent with the known facts does not, of course, establish its existence. If, however, it is regarded as worth consideration on this basis, we show that the share taken by degeneracy and imperfection would be as follows:

at 4·29° and 810 mm. $\quad p = n\kappa T\{1 - 0{\cdot}235 + 0{\cdot}031\}$,

at 5·20° and 1718 mm. $\quad p = n\kappa T\{1 - 0{\cdot}78 + 0{\cdot}10\}$,

where the two last terms represent respectively the effect of intermolecular forces and of non-Maxwellian distribution of molecular velocities.

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DISCUSSION

Prof. W. Wilson. I should like to ask the author how these low temperatures have been estimated. I had the impression that some equation of state such as that of van der Waals was assumed at the outset in order to determine them. It is interesting to note that Sommerfeld has recently applied the new statistical methods to the electron theory of the conduction of heat and electricity in metals. The classical theory gave far too big a specific heat to the metal. This is now corrected, I believe, by the degeneracy of the electron gas in the metal.

Mr J. H. Awbery. Professor Wilson has mentioned the use that Sommerfeld made of the theory of degeneracy. If I remember correctly, Sommerfeld supported his argument that electron gas should be degenerate at all temperatures known terrestrially, by comparing its degeneration quantitatively with that of hydrogen and helium as found experimentally (using the well-known molecular weight and temperature relation). His only exact evidence of degeneracy at that time was, of course, from specific heat data which, as mentioned in Mr Johnson's paper, give results that can only be attributed to degeneracy and not to imperfection. The fact that such uses have been made of the theory only increases the importance of an estimate such as Mr Johnson has made, and I should like to congratulate him on getting so much out of such scanty data.

Prof. J. E. Lennard-Jones. The new statistical theories developed by Bose and Einstein on the one hand, and Dirac and Fermi on the other, diverge most from classical statistics at low temperatures or high pressures, and calculation shows that it is only in the case of helium near its critical point that its behaviour is likely to be perceptibly different according as it obeys the classical, or the Bose-Einstein, or the Fermi-Dirac statistics. It is, therefore, a matter of great theoretical interest to determine the deviations of helium from the perfect-gas law near its critical point and to analyse the effect of the statistics and of the interatomic fields. Mr Johnson has made an interesting attempt on this problem, but there are one or two points in his treatment which seem to me to require investigation in any further work.

In the first place, the assumption is made that the effects of interatomic fields and of the statistics are simply additive, equation (2). As, however, the calculation of the effect of interatomic fields must itself depend on the statistics assumed, this simple separation of the two effects requires justification. In the second place, the author considers only the Fermi-Dirac statistics, whereas theory indicates that helium atoms should obey the Bose-Einstein statistics. It would add to the value of his work if the author could consider the effect of the latter statistics on helium near the critical point.

AUTHOR'S REPLY. Gas thermometry, at temperatures where imperfection and degeneracy are in question, is still accurate to a first approximation if the thermometric gas is at low pressure: this condition can be maintained while the gas whose temperature is measured may be at high pressure. Nevertheless, the question of Prof. Wilson and the first point raised by Prof. Lennard-Jones force us to recognise two or more stages of approximation in these problems. First assume that the gas laws hold and from the temperature readings deduce the amount of the consequences in imperfection and degeneracy. An equation of state can thence be constructed which may be used to correct the original temperature reading. With this corrected temperature a slight adjustment of the imperfection and degeneracy may then be made, according to the method, familiar in other branches of physics, of assuming that a certain error is zero in order to obtain a first approximation to its finite value, and using this to remove the assumption in a second approximation. Similarly, to the first approximation the imperfection and the degeneracy must be regarded as additive in order to determine their order of magnitude. Now that Leiden and Berlin are both obtaining data at the same low temperatures, the second approximation suggested by Prof. Wilson's remarks may soon be practicable, though I think the data are not yet accurate enough for the other second approximation which Prof. Lennard-Jones points out as ultimately necessary.

THE MAGNETOSTRICTION CONSTANT FOR ALTERNATING MAGNETIC FIELDS

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ABSTRACT. A magnetostriction constant K for the Joule effect is defined by the equation $p = KH$ where p is the alternating mechanical stress produced by a small alternating magnetic field H superposed on a steady magnetic field H_0. It is shown that the alternating intensity of magnetisation I produced by an alternating strain $\delta l/l$ is given by the equation $I = K\delta l/l$; K being the same constant in both equations. These equations are used to calculate the motional impedance of a laminated ring toroidally wound and vibrating in its fundamental radial mode. K has been measured for various values of H_0 with a ring of the alloy Fe 60 per cent., Ni 40 per cent. It is shown that the magnetostriction effect for steady magnetising fields can be inferred from the measurements with alternating magnetic fields, if the reversible permeability and the differential permeability of the alloy are known.

§ 1. INTRODUCTION

THE need for accurate and constant substandards of frequency has led to the use of oscillating valve circuits controlled by very resonant mechanical vibrators. The valve-maintained tuning fork[*] is available for sonic frequencies and the piezo-electric crystal [†] for radio frequencies. These vibrators are not convenient for frequencies between 5000 and 25,000 ~ and Pierce[‡] has developed the magnetostriction oscillator to cover this range. The theory of the valve-maintained tuning fork has been given by Butterworth[§] and others and that of the piezo-electric crystal by Dye[||] on the assumption that the vibrators can be considered as mechanical systems having one degree of freedom. This assumption is equally valid for the magnetostriction oscillator, and the theory appropriate to it must be closely analogous to that for the tuning fork and the piezo-electric crystal. It is essential in the three cases to know the nature and magnitude of the coupling between the mechanical vibrator and the electric circuit. This coupling is defined by the piezo-electric equations and the piezo-electric constant for the crystal and there must be similar equations and a similar constant for the magnetostriction oscillator. It is the object of this paper to define this magnetostriction constant and to show how it can be measured.

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[*] W. H. Eccles, *Proc. Phys. Soc.* **31**, 269 (1919).

[†] W. G. Cady, "Piezo-electric standards of high frequency," *Journ. Opt. Soc. of America*, **10**, 475 (1925).

[‡] G. W. Pierce, "Magnetostriction Oscillators," *Amer. Acad. Proc.* **63**, 1–47 (1928).

[§] S. Butterworth, "The maintenance of a Vibrating System by means of a Triode Valve," *Proc. Phys. Soc.* **32**, 345 (1920).

[||] D. W. Dye, "The Piezo-electric Resonator and its Equivalent Electrical Circuit," *Proc. Phys. Soc.* **38**, 399 (1926).

The magnetostriction oscillator used by Pierce* and Vincent† is generally a thin rod and the coupling to the electrical circuit is obtained by placing it in a solenoid carrying a small alternating current superposed on a steady magnetising current. The stresses arising from the magnetostriction effect set the rod into resonant vibration in its fundamental longitudinal mode, with a node at the centre and antinodes at the ends. The mechanical stress is greatest at the centre and falls to zero at the ends, with the result that the intensity of magnetisation must vary along the rod. There is a further complication due to the demagnetising effect of the magnetic poles appearing at the ends of the rod. The theory is much simplified by consideration of a magnetostriction oscillator in the form of a closed circular ring which is magnetised by a uniform toroidal winding. A ring has a mode of vibration in which all parts of it move radially with the same amplitude and phase and, if this mode is excited by the passage of an alternating current of suitable frequency through the toroidal winding, the magnetisation and the mechanical stress cannot vary along the circumference of the ring. There is, however, a variation in these quantities over the cross-section of the ring, since eddy currents are induced in it. If the ring can be laminated this variation can be much reduced and the stress and magnetisation become practically uniform throughout the material. Before investigating the motion of the ring and its reactions on the electrical circuit, it is necessary to define the magnetostriction constant in a form suitable for the study of alternating effects.

§2. THE MAGNETOSTRICTION EQUATIONS

The Joule effect is usually defined as the steady mechanical strain produced by the application of a steady magnetic field. This definition is inadequate for a study of the alternating strains produced by a small alternating magnetic field superposed on a steady magnetic field, for these strains depend on the frequency of alternation and the possible resonance of the vibrating body as well as on its magnetostriction properties. It is preferable to define the Joule effect as a relation between the alternating magnetic field and the alternating mechanical stress produced by it. If the laminated ring described above is imagined to be rigidly constrained so that no motion is possible, the alternating magnetic field from the toroidal winding will produce the same alternating mechanical stress at all frequencies and we may write

$$p = KH \qquad \text{......(1)}$$

where p is the stress in dynes/cm.^{2} H (or $H_0 \sin \omega t$) is the alternating part of the magnetic field in gauss, ω (or $2\pi \times$ frequency) is the pulsatance and K is a magnetostriction constant. There is a steady stress, produced at the same time by the steady magnetising field, with which we are not concerned. On removal of the constraints, the ring vibrates and we may regard the stress p as the force maintaining the vibration. The equation of motion can now be derived in the usual form for a vibrating system with one degree of freedom. Let ξ, $\dot{\xi}$ and $\ddot{\xi}$ represent the instantaneous outward radial displacement, velocity and acceleration at time t. Let r be the radius, σ the mean density and S the cross-section of the ring and let C be the velocity of sound in it and E the modulus of elasticity defined by the equation $C^2 = E/\sigma$. The circumferential stress p is equivalent to a total force, tending to expand the ring, of $2\pi Sp$ or $2\pi SKH$. The equation of motion is therefore

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* *Loc. cit.*

† J. H. Vincent, "Experiments on Magnetostrictive oscillators at Radio Frequencies," *Proc. Phys. Soc.* **41**, 476 (1929); *Nature*, **120**, 952 (1927); *Electrician*, **101**, 729 (1928); **101**, 11 (1929).

$$2\pi SKH = 2\pi rS\sigma\ddot{\xi} + k\dot{\xi} + 2\pi SE\xi/r \qquad \text{......(2),}$$

where the terms on the right-hand side represent in the order given the forces necessary to overcome the inertia, the damping and the stiffness. The symbolic solution of this equation for continuous and steady motion is

$$\dot{\xi} = \frac{2\pi SKH}{k + jw\,(2\pi rS\sigma - 2\pi SE/r\omega^2)} \qquad \text{......(3)}$$

where $j = \sqrt{-1}$. The displacement is a maximum, that is the ring resonates, at a pulsatance w_0 given by

$$2\pi rS\sigma - 2\pi SE/r\omega_0{}^2 = 0,$$

from which

$$\omega_0 = \sqrt{E/\sigma} \times 1/r = c/r$$

and

$$2\pi rf_0 = c \qquad \text{......(4).}$$

Resonance occurs at the frequency for which the wave-length of sound in the material is equal to the perimeter of the ring.

The velocity at resonance is

$$\dot{\xi} = 2\pi SKH/k = 2\pi SK\,(4\pi ni)/k \qquad \text{......(5)}$$

where n is the number of turns per cm. in the magnetising winding and i is the alternating part of the magnetising current.

The motion of the ring having been found, it remains to calculate the electromotive force induced in the toroidal winding as a consequence of the alternating Villari effect. An alternating mechanical strain ξ/r is accompanied by a change I in the intensity of magnetisation, where

$$I = K'\xi/r \qquad \text{......(6),}$$

K' being the magnetostriction constant for the Villari effect. The electromotive force v induced in the winding is equal to the rate of diminution of the magnetic linkage through the winding, that is

$$\begin{aligned} v &= -\,2\pi rnS \times 4\pi\dot{I} && \text{from (6)} \\ &= -\,2\pi rnS \times 4\pi K'\dot{\xi}/r && \text{from (6)} \\ &= -\,(2\pi)^2\,(4\pi)^2\,n^2S^2KK'i/k && \text{from (5)} \\ & && \text{......(7).} \end{aligned}$$

This expression for v shows that it is in opposite phase to the current i, and the electrical power required to maintain the vibration is therefore

$$(2\pi)^2\,(4\pi)^2\,n^2S^2K'Ki^2/k \qquad \text{......(8),}$$

which must be equal to the power consumed by the dissipative forces in the ring, that is, to $k\dot{\xi}^2$. Equation (5) shows that $k\dot{\xi}^2$ is equal to

$$(2\pi)^2\,(4\pi)^2\,n^2S^2K^2i^2/k \qquad \text{......(9).}$$

A comparison of (8) and (9) shows that $K = K'$ and therefore one magnetostriction constant is sufficient for the alternating Joule effect and the converse phenomenon, the alternating Villari effect. The magnetostriction equations become

$$p = KH \qquad \text{for the Joule effect} \qquad \text{......(10)}$$

and

$$I = K\xi/r \qquad \text{for the Villari effect} \qquad \text{......(11).}$$

This reciprocal relation between the two effects can be derived from a general theorem given by J. J. Thomson*; it has been proved here from first principles in order to present a physical interpretation of the phenomena associated with the vibrating ring.

An additional equation is required for the interpretation of the measurements described below. Equation (3) shows that the velocity of vibration is half that at resonance at two frequencies given by

$$\pm\sqrt{3}\,k = w\,(2\pi rS\sigma - 2\pi SE/r\omega^2) \qquad \text{......(12).}$$

If the ring is very resonant, we may write $w = \omega_0 + \delta w$ where $\omega_0/2\pi$ is the resonance frequency given by (4). On substituting this value above and neglecting $(\delta\omega/\omega_0)^2$, and higher powers, we find that

$$\delta w = \pm\sqrt{3}\,k/4\pi rS\sigma \qquad \text{......(13).}$$

Apart from the use of the reciprocal relation, this investigation differs from that of Pierce† in its definition of the modulus of elasticity. He expresses the magnetostriction force‡ in the form

$$p = aS\,(Bi + B'),$$

where Bi is the induction due to the periodic current i, B' is the induction due to the reverse magnetostriction effect, that is the Villari effect, and a is a magnetostriction constant for the Joule effect. The term aSB' is a force opposing the motion which is included in Young's modulus for the material and should not appear in his equation of motion§ as a term to be subtracted from Young's modulus. In fact a new modulus of elasticity which he introduces later‖ is actually Young's modulus; the discrepancy is of little practical importance and does not affect his subsequent analysis.

§3. THE MAGNETOSTRICTION CONSTANT

A quantity of the nickel-iron alloy containing 40 per cent. nickel was available in thin sheets 0·002 in. thick and, Pierce¶ having reported that it vibrated well, an attempt to measure its magnetostriction constant was decided upon. A number of circular rings of the alloy of 11·05 cm. and 8·55 cm. in external and internal diameter respectively, and a number of similar paper rings 0·0005″ thick were prepared. The metal rings were cleaned, covered on both sides with a thin layer of paraffin wax and piled alternately with the paper rings. The composite pile was clamped between two metal plates which were sufficiently hot to melt the wax and, after slow cooling, the finished laminated ring was taken from the press and superfluous wax cut away. The ring was placed on three rubber supports inside a circular wooden channel over which two toroidal coils of 227 and 215 turns were then wound (Fig. 1). The 227-turn winding carried a direct current supplied by accumulators through a large inductance and a variable resistance, and the other winding completed one arm of the alternating current bridge with which the measurements of motional impedance were made. The ratio arms of the bridge were non-inductive resistances of 100 ohms, and the remaining arm contained a calibrated inductance in series with a calibrated resistance, both continuously variable. Alternating current was supplied to the bridge from a coil inductively coupled to a 20-watt valve oscillator, the frequency of which was continuously variable. An earthed wire grid interposed between this coil and the valve oscillator screened the bridge from the electric field of the oscillator. The detector was a thermionic voltmeter reading from 0 to 1·5 volts and connected to the bridge through a tuned transformer. These simple arrangements were quite adequate for the measurements of impedances up to 100 ohms at 18,000~. The stray capacities which existed between the elements of the bridge were certainly less than 50 $\mu\mu$f. and, since this corresponds to a reactance of 178,000 ohms at the frequency used, the resulting error in the balance of the bridge was negligible.

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* J. J. Thomson, *Applications of Dynamics to Physics and Chemistry*, ch. 5. (Macmillan and Co., 1888.)

† *Loc. cit.* p. 35 and p. 40.

‡ *Loc. cit.* p. 35 (2).

§ *Loc. cit.* p. 36 (5).

‖ *Loc. cit.* p. 36 (7).

¶ *Loc. cit.*

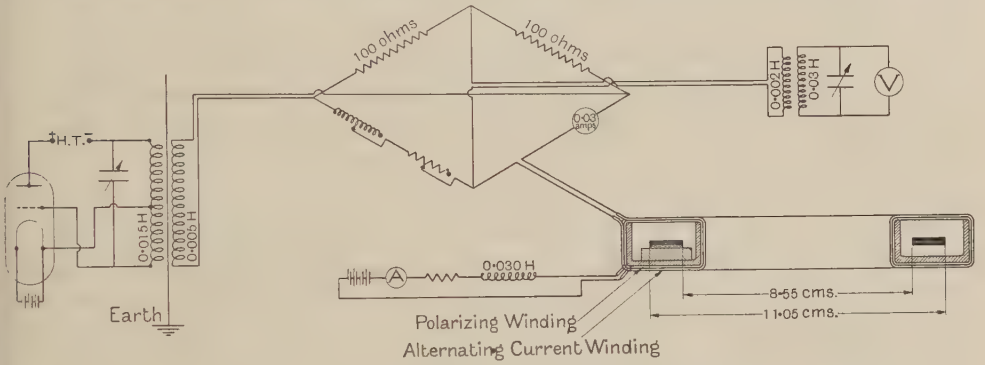

Fig. 1. Circuit for measurement of motional impedance.

Some care was required to bring the laminated ring to a reproducible magnetic state. The ring retained some of its magnetism when the magnetising current was broken and the intensity of magnetisation therefore depended upon its previous magnetic history. It was essential to demagnetise it before applying the required magnetising field. Residual magnetism was easily detected on the alternating bridge by the presence of motional impedance at frequencies in the neighbourhood of resonance and it was possible though not easy to remove it by momentary applications of a magnetising field. The sound emitted by the ring provided another sensitive test for the presence of residual magnetism, for this sound could be easily heard when any magnetism remained. The completely demagnetised ring vibrated at twice the frequency of the alternating current, and the sound emitted was, of course, far above the audible limit.

Measurements of the effective impedance of the alternating-current winding were made over a range of frequencies in the neighbourhood of resonance. Near the resonance frequency, a small continuous drift in the frequency of the valve oscillator, as the filament battery discharged, was accompanied by a comparatively rapid change in the impedance and, as two adjustments were necessary to obtain a balance, the latter was somewhat elusive. It was simpler to set the variable inductance to a suitable value and to obtain a balance by adjusting resistance and frequency. The measured impedances are plotted in Fig. 2 as points on a rectangular co-ordinate diagram with reactances as ordinates and resistances as abscissae. They lie closely on a circle and the resonance point lies near the end C of the horizontal diameter BC. If BA and BD are drawn so that

$$\angle ABC = \angle DBC = 60^\circ, \quad AB = DB = \tfrac{1}{2} BC,$$

we know, from the elementary theory of the circle diagram, that the amplitude of vibration at the frequencies corresponding to A and D is half that at the resonance point C. As will appear later, it is necessary to know the frequency increments corresponding to the arcs AC, DC in order to calculate the mechanical frictional forces opposing the motion of the ring. The frequencies marked on the diagram, which were measured with a resonance wave-meter, were not sufficiently accurate for this purpose. Small changes of frequency were more accurately found by measurement of the beat note between the valve oscillator supplying the bridge and another valve oscillator constant in frequency. The bridge was set successively to the impedances corresponding to the points A, C, and D, and after it had been balanced by an adjustment of frequency, the increments of frequency corresponding to the arcs AC, DC were found by measurement of the change in the beat note. It was found that the arcs AC, DC corresponded to a frequency change of $21{\cdot}5 \pm 1 \sim$.

The circle diagram of Fig. 2 provides a sensitive test for the presence of eddy currents in the laminated ring. Pierce* and Black† have shown that the effect of eddy currents is to depress the diameter BC, drawn through the resonance point, below the horizontal. The angle of depression was about 80° for a solid rod and about 40° for a nickel tube split longitudinally to reduce eddy currents. The motional impedance of the laminated ring at resonance is seen to be practically a pure resistance and it may be concluded that the eddy currents were too small to be appreciable.

The resonant ring thus complies with the assumptions in the theory given above, which can therefore be used to interpret the experimental results. The motional impedance at resonance corresponds to the diameter BC of the circle diagram and we may write from (7)

$$(2\pi)^2 (4\pi)^2 n^2 S^2 K^2/k = 15{\cdot}0 \times 10^9 \text{ e.m.u.} \qquad \text{......(14)}.$$

S, the cross-section of the ring, is 0·412 cm.^{2}, and n, the number of turns per cm. in the winding, is 7. The damping factor k is found to be 13,700 gm. sec. by substitution of the values $\sigma = 6{\cdot}9$ gm./cc., r 4·9 cm. and $\delta\omega = 6{\cdot}28 \times 21{\cdot}5$ in (13); and thence from (14) it appears that $K = 6{\cdot}28 \times 10^4$. There is a correction for the fact that the laminated ring contains 85 per cent. alloy and 15 per cent. paper and paraffin wax by volume. The value of K is an average over the cross-section and, since the paper contributes nothing to the magnetostriction effect, the value of K must be increased in the ratio of 100:85. The final result is therefore

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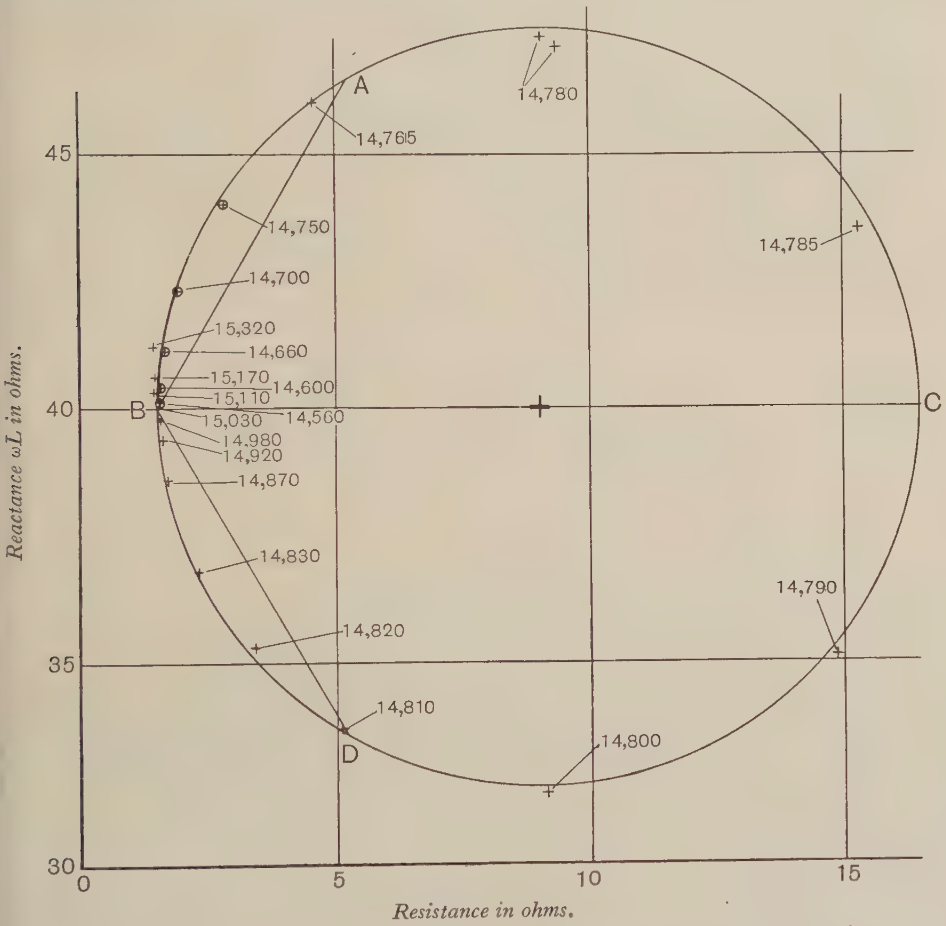
* *Loc. cit.* p. 37.

† *Loc. cit.* pp. 55, 61.

$$K = 7{\cdot}4 \times 10^4$$

for the nickel iron alloy containing 40 per cent. of nickel and permanently magnetised by a steady field of 14·8 gauss. Measurements with other magnetising fields are given in the table.



Diameter of circle 15·0 ohms. The amplitude is half that at resonance at 21·5 ~ above or below resonance. Small figures in circle are frequencies in cycles per second.

Fig. 2. Circle diagram for a vibrating ring.

Table.

Magnetising field	3·63	7·27	10·9	14·5	17·8	21·8	25·1	31·9
K for alternating fields ($\times 10^4$)	Very small	1·7	6·7	7·4	7·6	7·6	7·4	5·3
K for steady fields ($\times 10^4$)	Very small	17·3	53	42	30	23·5	19·5	12·2

§ 4. MAGNETOSTRICTION IN STEADY FIELDS

It can be shown that these measurements are consistent with the changes in length of the same alloy in steady magnetising fields given by Masumoto and Nara*. The alternating magnetic field superposed on a steady field produces changes of magnetic induction represented by an elongated hysteresis loop near the B H curve and inclined to it, the average slope of which is known as the "reversible" permeability†. The change of induction produced by a steady increment δH of a steady magnetic field H takes place along the B H curve and may be expressed in terms of the "differential" permeability which is the slope of the B H curve at the point corresponding to H. "Differential" permeability is always larger than "reversible" permeability, and therefore a steady increment of magnetic field δH produces a larger change of induction than an alternating field $\delta H \sin \omega t$. Now magnetostriction is a phenomenon resulting from the change of orientation of molecules by magnetisation and depends upon intensity of magnetisation. It follows that the change of length produced by a steady increment of magnetic field δH must be greater than that produced by an alternating field δH in the ratio of the "differential" susceptibility to the "reversible" susceptibility. This ratio was found by measurement of the slope of the B H curve for the differential susceptibility and by measurement of the effective inductance of the ring on the alternating current bridge for the reversible susceptibility. The magnetostriction constants for steady increments of magnetic field are obtained by multiplication of the constants for alternating magnetic fields by the corresponding ratio of susceptibilities, with the result shown in the second row of the table above. These figures are the slopes at a number of points of a curve connecting mechanical stress with the steady magnetising field producing it, and this curve can be found by graphical integration

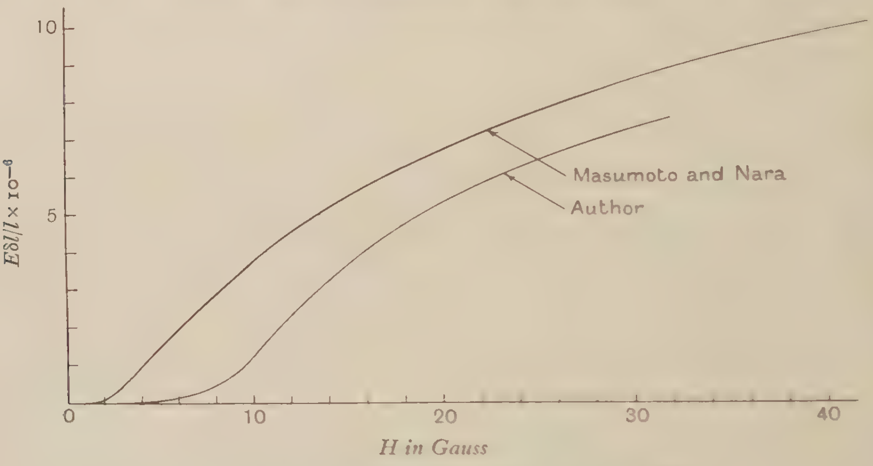


Fig. 3. Magnetostriction in steady magnetising fields.

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* Masumoto and Nara, *Sci. Rep. Tohoku Univ.* **16**, 338 (1927).

† Spooner, *Properties and Testing of Magnetic Materials*, p. 11. (The McGraw Hill Book Co. 1927.)

from the slopes. The result is shown in the lower of the curves of Fig. 3. The upper curve is obtained when the changes of length of the same alloy as given by Masumoto and Nara* are multiplied by Young's modulus. The shapes of the curves are similar and they are roughly asymptotic at the stronger magnetic fields. The differences at the weaker magnetic fields can be accounted for by a corresponding difference in the B/H curves. The alloy used by Masumoto and Nara was soft and that used by the author was somewhat hardened by cold working. Soft material magnetises more readily than hard material in weak magnetic fields and it should show a correspondingly greater magnetostriction effect in such fields. In strong fields, however, both materials tend to approach similar values of saturation induction and therefore should have similar magnetostrictive change of length. The relation between the magnetostriction curves for the two samples is in fact qualitatively the same as that between the B/H curves, and, moreover, the quantitative agreement at the higher magnetising fields is good.

It may be concluded that the method of measurement which has been described is practicable and sufficiently accurate for the comparison of the magnetostrictive effects in alloys subjected to alternating magnetic fields.

§ 5. ACKNOWLEDGMENTS

The author desires to express his indebtedness to the Admiralty for granting permission to publish this paper, and to Dr C. V. Drysdale, Dr E. P. Harrison and Dr A. B. Wood for their continued interest and encouragement.

DISCUSSION

Dr J. H. Vincent. I think that the Physical Society is to be congratulated on the presentation of this paper, which marks a distinct advance in the subject of the application of magnetostriction to the production of mechanical oscillations. I suppose that the formula given for the period of oscillation of an anchor ring in its fundamental radial mode is only approximate. The period of oscillation for an infinitely thin ring is the circumference divided by the velocity of sound in the material. If the anchor ring is supposed to have its orifice filled up with the same material we approach the case of a wire vibrating radially. Whenever a current flows in a wire of magnetostrictive material the radius of the cross-section must change in consequence of the magnetic field which the current produces. This should entail a number of interesting consequences, some of which might be capable of experimental demonstration. I should like the author to tell us a little more about the behaviour of the anchor ring when vibrating under the effect of alternating current in the absence of polarising current.

Dr D. W. Dye. I have read the present paper with much interest and wish to congratulate the author upon a difficult piece of work in a field not hitherto explored. I admire the author's courage in obtaining the results he has worked out, in spite of the admixture of paper, paraffin wax and nickel-iron alloy investigated. These must form a specimen of rather poor vibratory properties. There are three points to which I would like to draw attention. (1) In discussing the Young's modulus of the alloy the author states that Pierce should not have used the term aSB' as a term to be subtracted from Young's modulus. Pierce is quite consistent in what he has done; the difference between him and the author is simply one of definition of Young's modulus, which is different in the two cases. One is free to choose either. (2) The dimensions of the built-up ring, as shown in the diagram, indicate a ratio of 1 : 3 between outer and inner diameters of the ring. This will result in a corresponding ratio of magnetising fields between inner and outer edges. The results obtained are, therefore, those referring to an average magnetisation corresponding to this range of H. (3) The values obtained from the circle diagram and by the theory given pre-suppose a pure mode of radial vibration in which the stresses are purely compressional and tensile in the direction of magnetisation. From my experience of quartz specimens of all shapes and sizes I should not be at all surprised to find that a ring of the shape shown had overtone flexural modes such that the vibratory energy was divided up between two or more modes of vibration. This would profoundly affect the magnetostriction constant as here determined. In the analogous case of the quartz plate, the effective piezo-electric constant nearly always comes out at about one-third only of its theoretic value when the motion is complex, simply because energy is divided up between various modes, only one of which reacts piezo-electrically upon the maintaining circuit. I would suggest that a ring of greater thickness should be tried to see if the same result is obtained.

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* Masumoto and Nara, *loc. cit.*

Mr W. A. Benton. I consider the paper of quite outstanding interest and importance. There was one remark, however, made by the author in presenting his paper to which I felt exception might be taken. Mr Smith spoke of the "alternating current producing stresses unaccompanied by changes of dimensions in the magnetic material". It is difficult, if not impossible, to conceive of any stress not accompanied by strain, and, as a matter of fact, the mathematical reasoning would appear to hold good if small dimensional changes accompany the alterations in stress. It may interest Fellows to know that over 50 years ago Dr Du Moncel, and, I believe, others, described telephones based on magnetostriction effects. A rod of magnetic material attached to the centre of a diaphragm reproduced music and speech when acoustically modified currents circulated round the rod. It would not appear, however, that anyone has used the reverse, or Villari effect, as a telephonic transmitter. The very considerable changes of stress which could be produced by acoustic vibrations in a rod or wire made of a magnetic material especially susceptible to the Villari effect could be used to modify the strength of a current and thus form a transmitter.

Author's reply. The simple formula given in the paper for the fundamental mode of radial vibration of a ring is, as Dr Vincent points out, only approximate. An exact calculation which takes into account the finite radial width of the ring shows that the error involved in this approximation is negligibly small for the comparatively thin ring used in the experimental work. I am unable to give Dr Vincent any further information on the behaviour of a ring subjected to an alternating magnetisation only.

I agree with Dr Dye that the theory given by Pierce for the vibrating rod is accurate provided that Young's modulus for the material is suitably defined. Pierce does not, however, define this modulus in his paper, and the usual simple definition is therefore implied, whereas another rather complicated definition is required if his theory is to be consistent. There is no doubt in my mind that the predominant mode of vibration of the laminated ring is radial. The sound in air emitted by the vibrating ring is audible and it is possible to explore its surface with a small stethoscope. The sound intensity appears to be uniform over the whole of the outside and inside cylindrical edges. The flexural forces in the vibrating quartz disc arise from the fact that the diameter of the disc is usually large in comparison with the wave-length of sound in quartz at the frequency of the vibration. The expansions and contractions of the quartz in the plane of the disc, depending upon Poisson's ratio, are partially prevented by a lateral constraint from the inertia of the disc, and radial stresses are produced. These stresses are not quite symmetrical with respect to the quartz disc because the piezo-electric properties are not, in general, quite uniform throughout the disc, and stresses are therefore produced which can excite flexural modes of vibration. In contrast to the quartz disc, which is usually large in two directions in comparison with the wave-length, the laminated ring is small in two directions in comparison with the wave-length. Expansions and contractions in the directions of the thickness and the radius of the ring, depending upon Poisson's ratio for the material, are practically unconstrained. Moreover, the ring is built up from a large number of thin stampings assembled at random, and any appreciable variation of the magnetostriction constant along the circumference is improbable. It is difficult to imagine how any stresses can arise which might excite a flexural mode of vibration.

The conception of a material in a state of stress without strain, which Mr W. A. Benton finds difficult to accept, can perhaps be most clearly realised in the following case. Let a long thin rod of a material, having the property of magnetostriction, be magnetised so that its length and cross-section change. The rod can be restored to exactly the original dimensions by forces applied to its ends and surface, so that it is completely freed from strain. These forces are in equilibrium with the forces arising from stresses in the material, so that a state of stress without any strain whatever is produced.

FIELD INTENSITY MEASUREMENTS AROUND SOME AUSTRALIAN BROADCAST STATIONS

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ABSTRACT. The field intensities of wireless transmitting stations have been measured in order to determine some of the factors controlling the propagation of electromagnetic waves, and the effect of the surface conditions on their attenuation. A simple loop, condenser and valve voltmeter circuit has been employed, and careful tests show that this method is available for the measurements of intensities as low as 1 mv./m. The grid rectification voltmeter described is found to measure e.m.f.'s down to 0·01 volt, to be free from frequency error, reasonably stable in calibration, and not to absorb power from the circuit to which it is connected—the last being contrary to current theory. The advantages of this intensity set, which agrees over a wide range of field intensities with a Western Electric set, in respect to simplicity, convenience and portability are pointed out.

The field strength contours of three broadcast stations have been determined, from which the following conclusions have been drawn: (i) Very rapid attenuation of the signal is caused by Australian forest areas; this enormously curtails the areas for which a satisfactory service is provided. (ii) The effective conductivity of the various types of ground surface met with varies from 4×10^{-13} to $0{\cdot}07 \times 10^{-13}$ e.m.u., according to the number of trees in the areas covered. (iii) The use of a longer wave-length gives a marked increase of intensity at distant points beyond forest areas. (iv) For daylight transmission over sea water up to a distance of 85 miles, after the application of curvature-corrections to the intensity, Sommerfeld's formula is correct, to within the limits of experimental error. (v) The efficiency of radiation of the three aerials examined ranges from 48 to 60 per cent.

Finally, the limits of signal intensity required for satisfactory reception under Australian conditions are discussed, and it is suggested that atmospherics and other disturbances are less prevalent in Victoria than in Europe or America.

§ 1. GENERAL DESCRIPTION OF THE METHOD AND APPARATUS

If we have a loop aerial tuned by a condenser, then the ratio of the signal intensity to the voltage which it induces across the loop is constant for all values of the intensity. The value of this ratio depends on the geometry of the loop, its resistance and reactance, and can be determined for a given loop once and for all. Hence if this ratio is known, and if the induced voltage is measured at any point, the field intensity at that point can be deduced.

In the set used by the author the centre point of the loop was earthed, as shown in Fig. 1, in order to preserve the directional effect of the loop aerial by elimination of the "antenna-effect," and the induced voltage across one-half of the loop was measured with a valve voltmeter. The absolute value, E millivolts per metre, of the field intensity, was determined at this laboratory by an independent method, and the voltage V across half of the field loop was measured at the same place. Hence the ratio E/V for this loop was obtained; it was found to remain constant by similar comparisons carried out at regular intervals. Its value was

$$E/V = 86.$$

The valve voltmeter was calibrated after each reading of V in the field by means of a high-frequency oscillator, a vacuum thermocouple and a non-inductive resistance box.

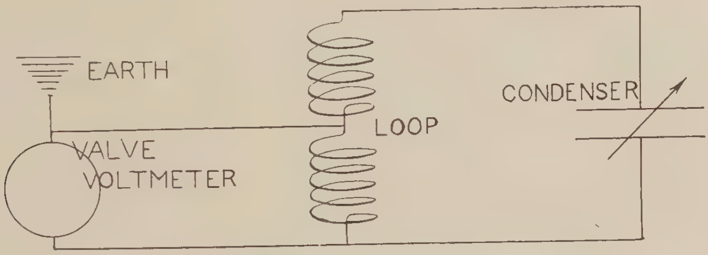

Fig. 1. Circuit diagram.

The measuring and calibrating equipment was mounted on a board in the back of a small motor car, while the fixed loop was carried in a case fastened on to the running board, and was erected on the side of the car for the taking of observations. Measurements made with the apparatus in the car, and on the ground with the car removed, showed that the car had no appreciable disturbing effect.

§ 2. DETAILS OF THE APPARATUS

The valve voltmeter. The circuit is shown in Fig. 2. The points of interest of the voltmeter are: (*a*) the absence of the artificial grid leak resistance, so that the grid of the valve is apparently "free." This greatly increased the sensitivity of the instrument and diminished the power absorbed. There was no sign of instability of the valve, probably because the conductivity of the base of the valve, the valve socket and the valve pinch prevents the grid from blocking. (*b*) The size of the grid condenser, viz. $0{\cdot}002\,\mu$f. (*c*) The balancing-out device for eliminating the steady plate current from the microammeter. This affords a very sensitive means of adjusting the zero of the instrument to any point of the scale. (*d*) The loop tuning condenser; as neither set of plates was at earth potential, in order to avoid body-capacity effects it was necessary to enclose the condenser and voltmeter in an earthed shield connected to the middle point of the loop. Although the batteries and the tuning condenser handle were not shielded, there was no disturbing hand-capacity effect during tuning. The condenser was a double-spaced transmitting condenser of maximum capacity $85\,\mu\mu$f.

The small link between the earth and grid terminals of the voltmeter was found to be of great practical convenience when frequent interchanging of the connections was necessary. The grid terminal was very susceptible to large fluctuations of potential while connection was being made, causing momentary paralysis of the valve.

Characteristics of a voltmeter. These are: (i) sensitivity and range, (ii) frequency error, (iii) wave form error, (iv) power used, and (v) stability of calibration.

The relative importance of these characteristics will depend on the circuit to which the voltmeter is connected and the quantity that is to be measured. When a small high-frequency current is to be measured by means of the voltage across an inductance tuned to resonance with the applied e.m.f., it is essential that the voltmeter should absorb no power from the circuit. In the measurement of field intensities by the method outlined above, any power taken by the voltmeter will appear as an increase in the resistance of the loop.

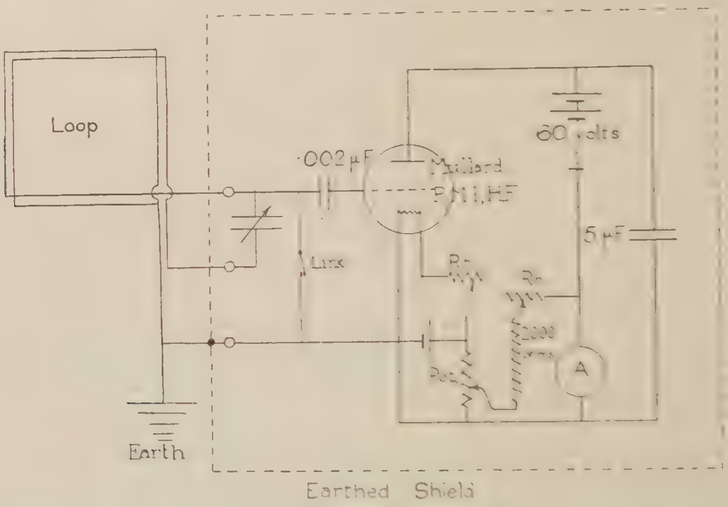

Fig. 2. Valve voltmeter circuit.

Performance of the grid-type valve voltmeter. (i) Sensitivity. With a portable microammeter having a sensitiveness of 10 mm. per microamp., the lowest measurable voltage was 0·01 volt (deflection about 2 mm.) corresponding to a field intensity of 0·86 mv./m. This sensitivity was obtained by the correct choice of valve (Mullard P.M. 1 H.F.), the operating voltages, and the absence of an added grid leak. With a grid-leak resistance of 10 megohms the sensitivity was considerably diminished, especially for small voltages where the extra sensitiveness was most required. (ii) Frequency error. If the voltmeter is to be independent of frequency, the reactance of the grid condenser must be small compared with that of the grid-filament capacity of the valve and its socket. Taking the latter as not greater than 30 $\mu\mu$f., the grid condenser was fixed at 0·002 μf. To check this point, the voltmeter was calibrated at 600 kc. and 900 kc. and the agreement was within 1 per cent. In the field the calibration was carried out at 750 kc., which is about the middle of the broadcast band under investigation. If the grid condenser was larger than 0·01 μf., the grid took an appreciable time to reach its final steady potential under the action of the incoming signal, and there appeared to be no advantage gained by the use of a larger value than that adopted. (iii) Wave form error. In order that the voltmeter reading should be independent of the wave form of the applied e.m.f., it is necessary that the rectified current be proportional to the square of the applied voltage. For the calibration curve below 0·1 volt this condition is fulfilled, as can be seen from Table 1, which is taken from a typical curve.

Table 1

Rectified current	0·2	0·67	1·45	2·75	4·3	6·15	8·4	11·2	17·1 μa.
Applied potential	0·01	0·02	0·03	0·04	0·05	0·06	0·07	0·08	0·1 volt
$k = \frac{\text{Current}}{(\text{potential})^2}$	44·7	40·8	40·1	41·5	41·5	41·3	41·4	41·9	41·4

Above 0·1 volt the curve departs from its parabolic form and it becomes straight from 0·15 up to 0·4 volt. Hence for applied voltages below 0·1 volt, the calibration is independent of wave form, while above this point it will depend on wave form. This characteristic is important when the induced voltage due to a modulated signal is being measured. Above 0·1 volt, there were considerable fluctuations of the voltmeter reading due to the modulations of the signal, while below this figure there was practically no sign of such modulation, even though the deflections of the microammeter were as large as 80 mm. As only the carrier-wave intensity was required, it was necessary to wait until the modulation ceased. The short period of the instruments used made this cessation easy to detect, a silent period of 2 seconds being enough to allow the pointer to take up its position for the unmodulated signal. No special precautions were taken to get a pure wave-form for calibrating the voltmeter except that the coupling between the oscillator and the tuned circuit containing the thermocouple and resistance box was kept loose. (iv) Power used. As was mentioned above, the power absorbed by the voltmeter should be small and independent of the applied voltage. A valve which uses the curvature of the grid-current grid-potential curve for rectification must absorb some power, because the action of the rectifier depends on the presence of grid current. In broadcast receivers, the grid condenser is usually connected to the positive terminal of the filament, with a resistance of 1 to 3 megohms across the condenser, and such an arrangement increases the resistance of the input circuit enormously, necessitating the use of reaction to counteract it. This damping of the input circuit increases as the input voltage decreases, and largely counterbalances the gain in sensitivity due to the grid condenser and leak. In the voltmeter described above these defects are reduced to a minimum, while the sensitivity is increased, by the omission of a grid leak. This arrangement is unsuitable for the detector valve in a broadcast receiver. No appreciable effect of the voltmeter on the input circuit could be detected by the following test. Two instruments of identical construction were connected in parallel across a loop tuned to resonance, and a small voltage was induced in the loop by a local oscillator. Upon the switching out of the filament in one voltmeter, no change was detectable in the reading of the other meter. This test repeated for a series of voltages gave no evidence of power absorption. This only shows that the grid voltmeter has the same effect on the loop circuit whether the valve is alight or not, but bridge measurements at 1000 ~ showed that in both cases the input of the voltmeter with tuning condenser behaved as a nearly perfect condenser of about 30 $\mu\mu$f. capacity. The same test, repeated with an anode-bend valve voltmeter and the grid voltmeter, showed that the former instrument absorbed power. Evidence to the same effect had previously been obtained by other means. (v) Stability of calibration. The voltmeter was calibrated after each reading of the voltage, so that stability of calibration, although it obtained fairly well, was never assumed. The variations during a day were somewhat irregular, but the constancy was usually within 10 per cent. The calibration curves for a single valve over a period of several months were all of the same form, showing an equal percentage change over the full range. Hence one or two points on the curve served to fix the whole of it with sufficient accuracy for the deduction of the signal intensity.

The microammeters. For applied voltages above 0·08 volt, a Paul unipivot galvanometer (1·8 μa. per division) was used; this instrument when shunted also measured the steady plate current. For lower voltages, a portable reflecting Tinsley galvanometer (10 divisions per μa.) was used. The shortness of the period of these instruments made them very suitable for the work. These two galvanometers were mounted together in a box which was suspended by six small coil springs and rested on rubber sponges inside another box. The efficacy of this suspension was shown by the fact that in over 5000 miles of travel, for a large part on rough and unformed country roads, no trouble was experienced with this reflecting instrument. No special mounting was necessary for the rest of the apparatus.

The microammeters showed no instability due to the absence of an artificial grid leak, but there was usually present a gradual creep of the zero of the voltmeter if the batteries and valve had not reached their steady state. No difficulty was experienced in allowing for this creep of the zero, unless it exceeded the rate of 1 mm./sec.

Calibration of the voltmeter. The voltmeter was calibrated by a potential measured, as in Fig. 3, by means of a thermocouple and a non-inductive resistance box calibrated with high-frequency current. Care was taken to minimise stray capacity effects and good agreement was obtained between a high-frequency calibration and a 50 ~ calibration of an anode-bend voltmeter designed to be independent of frequency.

The field loop. In order that the ratio of field intensity to induced voltage, i.e. E/V, may remain constant, it is essential that the loop aerial used in the field should retain its shape and size, and also that its high frequency resistance should remain constant. While it was desirable to reduce this as far as possible to get a more efficient aerial, it was even more important to secure permanence, even at the expense of efficiency. It was therefore decided that solid copper wire should be used instead of Litz wire. The loop was a square solenoid (side 65·1 cm.) of 18 turns of no. 24 gauge enamelled copper wire, spaced 0·2 in. apart on thin bakelite strips. This construction gave an exact centre point, and should diminish any possibility of change in resistance during the field work which was carried out during fine weather. In the course of 3 months no evidence was obtained for any change as great as 2 per cent. An occasional tightening of the loop was the only attention required.

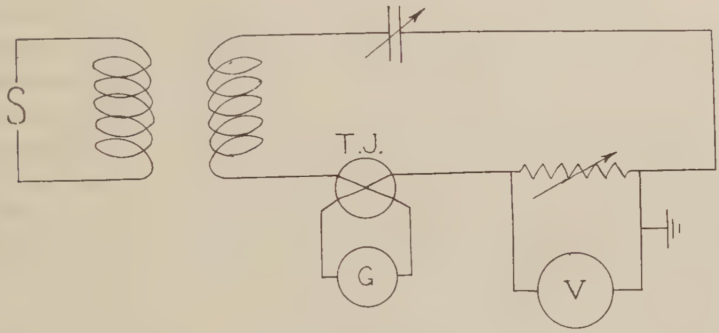

Fig. 3. Calibration circuit.

§ 3. DETERMINATION OF THE RATIO E/V FOR THE FIELD LOOP

The relation between the field intensity E mv./m. and the voltage V millivolts induced between the ends of a tuned loop is

$$V = 2\pi A n \omega L E / R \lambda$$

where A is the area of each turn in metres^2,
n the number of turns,
λ the wave-length in metres,
$\omega/2\pi$ the frequency,
L the inductance in henries,
R the resistance in ohms.

All of these quantities can be measured. From the above relation it would seem to be possible to get the value of E/V for the whole loop, and hence for half the loop, but the voltage across the half-loop was found to be more than half that across the whole loop. This result was due to the circumstance that the natural frequency of the loop was comparable with that of the stations under investigation, and difficulties were experienced from the same cause in the measurement of the high-frequency resistance of the loop. While approximate corrections for this self-capacity effect could have been calculated, it was decided to measure the absolute value of the intensity with another loop, at the same time measuring the voltage across half the field loop. Hence the ratio E/V was at once obtained. It was $E/V = 86$. This ratio was checked at intervals during the field work by the same method.

§ 4. DETERMINATION OF THE ABSOLUTE VALUE OF THE FIELD INTENSITY

A standard loop was constructed for this purpose and was kept at this laboratory, the intensity being calculated from the voltage induced across the whole of the loop as measured by the valve voltmeter. It was a square (side 132·3 cm.) of 4 turns of $\frac{5}{8}$ in. woven copper braid, spaced at 4 cm. apart. The inductance and resistance were determined separately and the value of $\omega L/R$ was deduced for a wave-length of 371 metres. This ratio was checked by the capacity-variation method. A special set of resistances was constructed and the measurements were repeated with both the voltmeter and a vacuum thermocouple as the detector in all determinations of R, L, or $\omega L/R$. The following values were obtained: $L = 51{\cdot}9$ microhenries, $R = 0{\cdot}913$ ohms when $\lambda = 371$ metres, so that $\omega L/R = 287$.

By capacity variation $\omega L/R = 291$. The mean value of the step-up is therefore 289, the high value being due to the low resistance of the copper braid. With this loop, the absolute value of the intensity is considered to be accurate to 1 per cent.

To check the absolute value of the intensity as determined with the standard loop, simultaneous readings at the University were taken with it and a Western Electric field intensity set, the property of the Research Laboratory of the Postmaster-General's Department. The mean value of the intensity determined by both sets was 55 mv./m. The standard loop was similarly calibrated for a wave-length of 484 metres, corresponding to that of the other local broadcast station, 3AR Melbourne. A comparison with the Western Electric set gave an intensity of 50, as against 51 mv./m. read by the standard loop. The ratio of E/V for the field loop for this wave-length was $E/V = 107$.

To test if E/V for the field loop was independent of E, a series of readings over a wide range of intensities was taken in conjunction with the Western Electric set. With the value $E/V = 86$ obtained above, Table 2 gives the results obtained:

Table 2

W.E. set	59·5	30·5	13·4	10·4	7·5	2·8	2·15	1·46	0·55 mv./m.
Valve voltmeter set	59·5	30·5	14·0	10·4	7·6	3·0	2·4	1·5	0·5 mv./m.

It therefore appears that this method of measuring intensities is as accurate as is required in this class of work.

§ 5. MERITS OF THE VALVE VOLTMETER FIELD INTENSITY MEASURING SET

The apparatus described will measure satisfactorily field intensities as low as 1 mv./m. This is a field too small for satisfactory reception according to English and American authorities.

Advantages. The following advantages are claimed for the set: (i) Simplicity of construction. This is apparent from the details already given. The voltmeter and oscillator are made of inexpensive wireless equipment, and the high-frequency decade resistance box, vacuum thermocouple and galvanometers are all easily obtainable. No elaborate shielding, such as is essential on sets making use of loop-injection methods, is required, and considerable weight is thus saved. Other difficulties, e.g. the attenuation box or the carefully balanced input circuit, are also avoided. (ii) Simplicity of operation. The loop tuning condenser is the only tuning control, and with a vernier drive this is easy to operate. The voltmeter deflection for several broadcasting stations can be quickly determined, and a single frequency of calibration serves for all of them. This is an important simplification for rapid and accurate work. (iii) Speed of work. The whole of the field work, including driving, observing, booking and reduction of the results, was done by the author. The observations used for plotting the contour map were obtained in 25 days in the field. In this period 497 individual observations were taken on two stations during their normal daylight working hours, no special transmissions being arranged. The area surveyed was over 10,000 sq. miles, and the distance travelled was 3067 miles. The time taken in an observation on two stations was about 7 minutes, from stopping to starting again. During the 9 months that the apparatus has been in use, not a single delay due to failure of any part of it has been experienced. (iv) Portability. The loop and voltmeter alone can easily be carried by one man without a car, and this enables the apparatus to be set up in places to which a car cannot get. This property enabled the detailed effects of hills on the field intensity to be studied.

Disadvantages. The great disadvantage of this type of set is its inability to measure very weak intensities. The rectifier used should obey a square law, i.e. the rectified current should be proportional to the square of the applied voltage, which holds for this instrument below 0·1 volt. For an intensity of 1 mv./m. the deflection is about 2·5 mm., and half of this intensity would only give 0·6 mm. By slight alteration of the construction of the loop, however, without increase of its dimensions, the ratio E/V was reduced from 86 to 63 and a slightly more sensitive detecting valve (P.M. 12) has been added. The deflection for 1 mv./m. is now 7 mm. No simple way of increasing the range of the instrument further, while still retaining its portability and other advantages, has been found. As there is only a single tuned circuit, lack of selectivity would also be a difficulty under certain conditions.

§ 6. ROUTINE IN THE TAKING OF AN OBSERVATION

For taking an observation, open sites likely to give the undisturbed value of the intensity in that locality were always chosen. The chief disturbing factors were: (i) Telephone wires, especially if present in large numbers. In one case a bank of 50 wires diminished the field intensity for over 50 yards on each side of the line. In another case, two wires doubled the intensity directly under them and produced a directional error of nearly 50°. The field was normal again 15 yards on either side of the wires. (ii) Crests of hills. On a steep slope facing toward the broadcast station, the intensity was always greater than at the foot of the hill or at a distance behind the crest. (iii) In timbered country, an open place at least 50 yards from any large tree was always selected. The site having been selected, the car is turned side-on to the direction from which the signal is coming. The valve, which is carried separately, is inserted into the voltmeter, and the loop is set up. The loop is tuned and rotated to give the maximum deflection on the microammeter, which is noted. The voltmeter terminals are then short circuited with the link, the loop terminals are disconnected, and short leads from the resistance box are connected. The link is opened, the local oscillator started and the current is adjusted to give a convenient reading on the thermocouple metre. The voltmeter deflection and thermocouple currents for known resistances are noted, and the calibration curve is plotted. In the field this curve is only taken twice a day. For most of the observations, a single point near the observed voltmeter reading or two points on either side of it are sufficient, on account of the similarity of all the curves, as has already been explained. From the calibration curve the voltage due to the incoming signal is found and the intensity is at once deduced.

§ 7. RESULTS OBTAINED IN VICTORIA ON 3LO AND 3AR MELBOURNE

Field intensity map of 3LO *Melbourne*. The observed intensities were plotted on a map, and the 15, 10, 5, 2·5 and 1 mv. m. contours were interpolated, as shown in Fig. 4. Only three observations out of 330 that are plotted are not in agreement with the contours as shown; two of these were suspected at the time of observation to be affected by ground contours but they were taken at the only available sites.

Effects of trees and mountains. The shape of the contours indicates that very marked absorption of the signal takes place in certain directions. In every case, these areas are heavily timbered and are usually associated with mountains or steeply sloping hills. In Victoria mountains and forests usually occur together, and it has not been possible to discriminate completely between them as to the cause of the absorption. In a few cases where the timber is dense while the country is only undulating, the rapid absorption has still been found, but it is not possible to state the effects of bare mountainous country. Only in a few directions does the transmission take place over fairly clear ground and the area given a satisfactory service is far less than was expected from the extent of the 10 mv. m. contour which was located before the present apparatus had been developed. In heavy timber the intensity falls from 10 mv. m. to 1 mv. m. in a distance of only 10 miles. Such areas exist to the east behind Mount Dandenong, to the north-east behind Whittlesea, and to the north-west in the Daylesford forest. Barfield in England* found that the signal was attenuated more rapidly when it passed over forest areas than when it passed over clear country, but in no areas were the effects so pronounced as in the present observations. This discrepancy is due to the difference in size and the number per unit area of the trees, in the two cases, for in the areas dealt with in the present paper the forests cover many hundreds of square miles,

* *Journ. Inst. E. E.* **66**, 204 (1928); **67**, 253 (1929).

while the density of the trees usually exceeds 10,000 per km². In many cases the number is probably greater than 30,000.

The height and leafiness of the trees was found to play an important part in the attenuation. Thus, in the areas mentioned, the trees are mostly over 100 feet high and have a fairly luxuriant foliage. In contrast to this, there is an area about 40 to 50 miles west of 3 LO covered with small trees, between 20 and 40 feet high, and with comparatively little foliage. The effect of this belt of timber is plainly seen in the form of the contours in this direction, but it is not so marked as in the former areas. This very rapid attenuation of the signal as it passes over forest areas explains the difficulty experienced in receiving Melbourne broadcast stations,

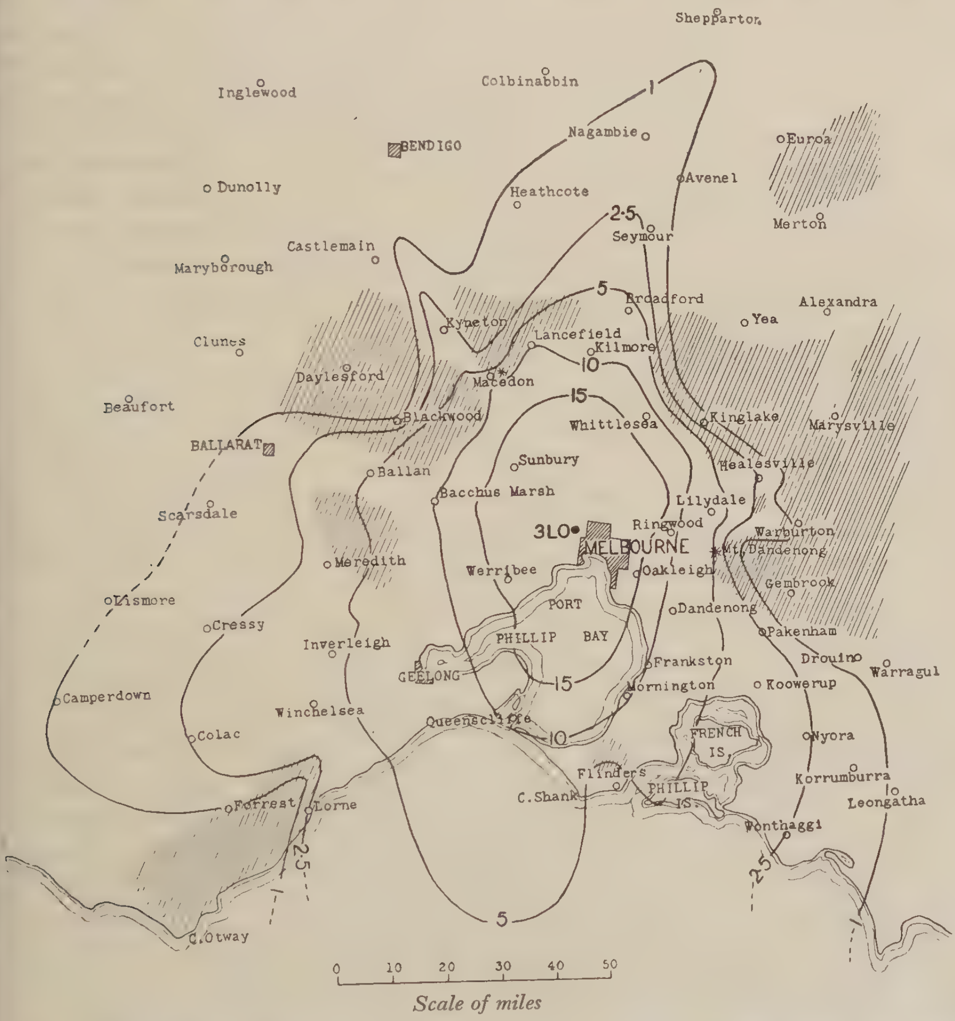

Fig. 4 Field intensity contours of 3 LO Melbourne, wave-length 371 metres, power 5000 watts.* Shaded areas represent mountains and forests.

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* This is the power in the plate circuit of the power amplifier coupled to the aerial.

in the eastern parts of Victoria, where, during daylight, the signal is very weak but of uniform intensity, while, after dark, fading and distortion are so bad as to spoil the programmes completely. These reception difficulties start at Warragul, about 60 miles east of Melbourne, where the daylight intensity is considerably less than 0·5 mv./m. The conditions existing in Gippsland are exactly similar to those in certain areas around New York City and investigated by Bown, Martin and Potter*. In the latter case the rapid absorption due to steel-framed buildings in the city near the transmitter produced areas within 35 miles of the station in which the daylight intensity was as low as 50 μv./m., and within these areas fading and distortion were very pronounced at night. It now appears that large areas of mountainous and timber country can produce exactly similar conditions at comparatively short distances from the station.

The land area within the 1 mv./m. contour is 9400 miles^2. If the ground had all been of average conductivity, so that the 1 mv./m. contour was a regular curve with its limits at the existing contour to the south-east, north and south-west, the land area within it would be over 20,000 miles^2.

Determination of the ground conductivity for various surface conditions. This can be deduced from the field intensity contours by applying Sommerfeld's theory of ground absorption†. The convenient curves published by the British Broadcasting Corporation‡, which give Sommerfeld's formulae in graphical form, make it possible to write down the intensity at a point distant from a transmitter, given the ground conductivity; or, as in the present case, to reverse the process. No curvature corrections have been applied, and it is considered safe to use the curves up to 100 km. only. The three directions in which the ground conditions are approximately uniform are: (i) South-west. The country is as uniform as could be desired, as there are but a few low hills and very few trees. The five contours (15, 10, 7·5, 5 and 3 mv./m.) agree excellently with those predicted from Sommerfeld's theory for a conductivity of 4×10^{-13} e.m.u. or a resistivity of 2500 ohm-cm. This value is taken as the true "bare ground" conductivity for Victorian conditions. The corresponding value determined by Barfield and others for English conditions is $2{\cdot}5 \times 10^{-13}$ e.m.u. (ii) South-east. The country is flat but there is a good deal of low scrub and some light timber is present in parts. The ground for a large part has a tendency to be swampy in winter. The five contours (15, 10, 7·5, 5 and 2·5 mv./m.) agree fairly well for a conductivity of $3{\cdot}3 \times 10^{-13}$ e.m.u. (iii) North. The country tends to become hilly and timbered at Kilmore and Broadford, 35 miles north of 3LO. There is however a narrow pass in which the signal is not rapidly attenuated. The country then becomes flat again, but a considerable amount of scattered timber still remains, a rough estimate giving from 200 to 500 trees per km.^{2} The country is all comparatively dry, and the conductivity deduced from the 10, 7·5, 5 and 2·5 mv./m. contours is 10^{-13} e.m.u. In heavily timbered and mountainous country, the effective conductivity deduced is as low as $0{\cdot}07 \times 10^{-13}$

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* *Proc. Inst. Rad. Eng.* **24**, 57 (1926).

† *Annalen der Physik.* **28**, 665 (1909).

‡ *The Service Areas of Broadcasting Stations*, by P. P. Eckersley.

e.m.u., which is only 1/50 of its value under favourable conditions. The value given by Eckersley* for this type of country is $0{\cdot}05 \times 10^{-13}$ e.m.u.

All these values are deduced from the contours which are interpolated between readings of the intensity taken at suitable points. More accurate determinations could be made from special observations taken at selected stations, with checks on the station output, but in only a few directions is the ground of sufficient uniformity to justify such an undertaking. In the most favourable direction, viz. south-west, the five points taken along a radial line agree excellently with the value stated.

Recent determinations of ground resistivity made by Mr Bruckshaw, of the Imperial Geophysical Experimental Survey, by Gish and Rooney's method give the value 1500 to 2000 ohm-cm., which agrees satisfactorily with the "bare ground" conductivity determined above.

Table 3

Type of country	Distance *d* in miles	Product *Ed*	
		3 LO $\lambda = 371$ metres	3 AR $\lambda = 484$ metres
Flat with not many trees S.W. and S.E. of 3LO ...	25	100	100
	75	65, 55	78, 75
Undulating with some timber N. of 3 LO	25	100	100
	75	30	55
W. of 3 LO	30	100	100
	60	40	62
Sea water	25	100	—
	73	87	—

Shielding effects of the masts at the transmitting station. In the field strength contours around 2 LO London, as shown by Barfield and Munro†, large crevasses occurred owing to absorption in the masts supporting the transmitting aerial. There is no sign of similar effects in the case of 3 LO Melbourne. The masts at the latter station are 200 feet high and are earthed at their bases. The aerial is in the form of Γ with the horizontal "roof" pointing to the north.

Effect of change of wave-length on the intensity at distant points. Information on this matter was obtained from observations on 3 AR Melbourne on a wave-length of 484 metres. This station was "on the air" only during the morning, and results for it are not so complete as for 3 LO Melbourne. The general form of the contours is fairly reliable, and Table 3 summarises the results of 160 observations. In it is shown the decrease of the product intensity × distance with increase of distance for various types of country. For purposes of comparison, all the values are so reduced as to make the product exactly 100 at the shorter distance.

The decrease of *Ed* in the above table will partly be due to the curvature of the earth, the effect of which at more than 50 miles from the transmitter is appreciable. It has not yet been possible to separate the losses into true-absorption and curvature losses.

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* Eckersley, *loc. cit.* p. 8.

† Barfield and Munro, *loc. cit.*

The aerial system at the station 3AR Melbourne does not even approximate to a vertical uplead with a horizontal roof, there being a certain amount of energy directed upwards at an angle of from 30° to 60° in certain directions. The observations contain evidence that some of this energy is returned at distances greater than 40 miles, and so it is not possible to apply Sommerfeld's formulae to deduce the ground conductivities from this station.

Table 4 gives the decrease of intensity due to a small extent of heavily timbered and hilly country.

Table 4

Width of timber in miles	Decrease of intensity	
	$\lambda = 371$ metres	$\lambda = 484$ metres
6	12·4–1·6 mv./m.	14·5–5·7 mv./m.
5	8·6–2·0 ,,	8·0–3·6 ,,
6	7·5–3·7 ,,	6·6–4·1 ,,
13	10·5–2 ,,	10·5–3·6 ,,

All the quantitative values for the conductivity of the various types of ground surface dealt with above and the variation of the attenuation with wave-length are deduced from values of the intensity extending over a wide area of country, in order to obtain the general features of the transmission. More detailed investigation of these values will be undertaken.

Transmission over sea water. In order to investigate the decrease of intensity due to the effects of the curvature of the earth alone, a series of readings was taken over sea water where the surface has a uniform high conductivity (10^{-11} e.m.u.). The readings were taken during the daylight part of the journey from Melbourne to Burnie, Tasmania, which extended to 90 miles from Melbourne. The motion of the boat made the readings of the unipivot galvanometer uncertain of 1 mv./m. in all the readings at distances of more than 40 miles. A large number of readings was taken and the time for each was recorded. The position of the ship at that time was computed from its measured speed over a distance of 30 miles before entering open water. The results are marked with centred circles in Fig. 5, in which $\log Ed$ is plotted against d.

If the intensity is corrected for curvature from results given by Ratcliffe and Barnett* for similar wave-lengths, the points marked with crosses are obtained; they lie about a horizontal line. This result indicates that there is no attenuation due to the finite conductivity of sea water up to a distance of 85 miles, and is in agreement with Sommerfeld's formula for transmission over a plane surface of conductivity 10^{-11} e.m.u., which gives the departure from the horizontal line at that distance as less than 2 per cent. To test this point further, readings were taken on 3LO and 3AR on both sides of Port Phillip Bay at the edge of the water. Simultaneous checks on the station output were obtained by separate intensity measurements at a fixed point, and large-scale ordnance maps were used to make sure that the measuring sites were in line with the transmitter. Three separate lines from each of 3 LO and 3 AR were taken, and the length of sea water varied from 31·6 to 35·1 miles.

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* *Proc. Camb. Phil. Soc.* **23**, 288 (1926).

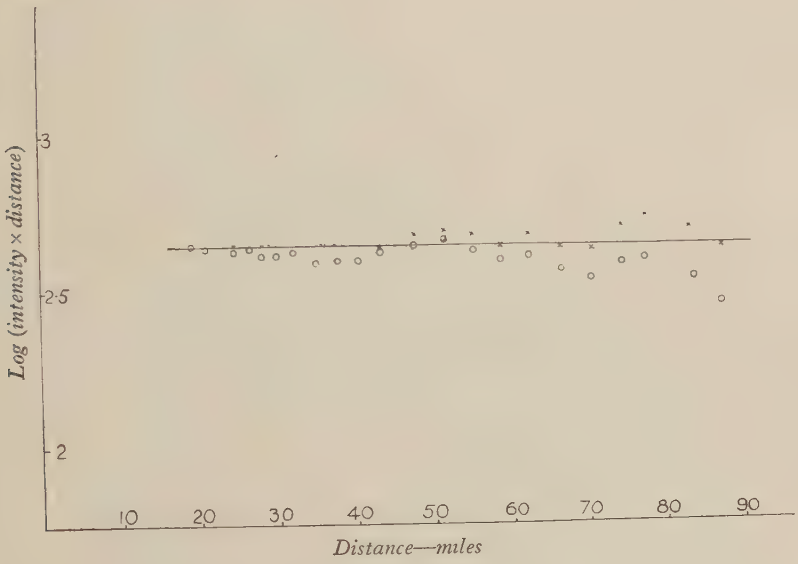

Fig. 5. Transmission over sea.

In every case the product *Ed* for $\lambda = 371$ metres showed a decrease, the mean being 5 per cent., while for $\lambda = 484$ metres, the product increased, the mean value of the increase being 4 per cent. This increase is attributed to the return of the energy initially radiated upwards from the aerial. It may be mentioned here that a slight increase in the product *Ed* with a small increase in *d* has been found frequently, even in the case of flat land transmission, for both stations.

§ 8. RESULTS OF OBSERVATIONS ON 7ZL HOBART

The wave-length of this station is 516 metres and its rated power is 800 watts. The field-intensity contours for 10, 7·5, 5, 3 and 1 mv./m. are shown in Fig. 6. These are drawn in from about 140 observations. Nearly the whole of Tasmania consists of wooded hills and mountains, and the effect of these is seen plainly in the form of the contours. The effect of Mount Wellington (4000 ft.) to the west of Hobart is very marked. Only to the north of the station does the transmission take place over undulating country without very much timber. The conductivity of the ground in this direction is $= 0{\cdot}5 \times 10^{-13}$ e.m.u. Tasmanian observations are in complete agreement with the conclusions stated for above Victoria.

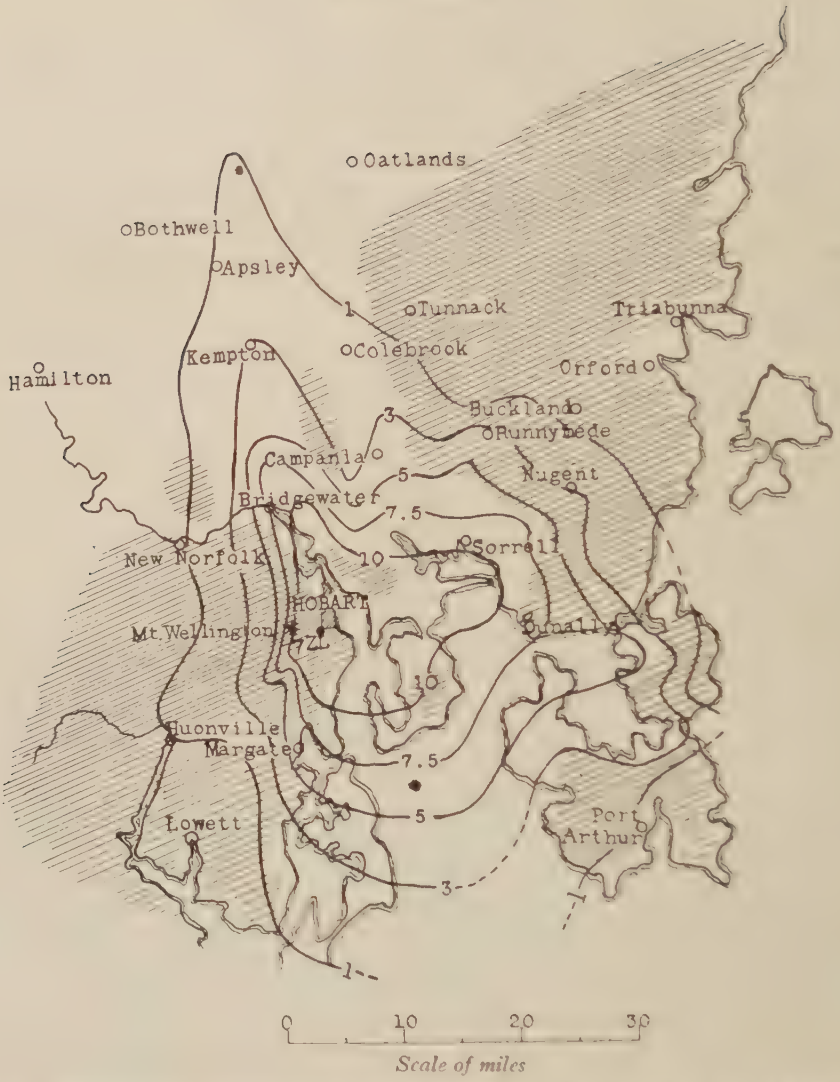

Fig. 6. Field-intensity contours of 7 ZL Hobart. Wave-length 516 metres, power 800 watts.

§ 9. EFFICIENCY OF THE AERIALS OF 3LO, 3AR AND 7ZL

The intensities at points on a circle of small radius d km. from the transmitter are averaged, giving the mean intensity at that distance, for which loss due to absorption will be negligible. If E volt metre is this mean intensity, the power radiated by the aerial is

$$P = E^2 d^2 10^5/9 \text{ watts.}$$

Table 5 gives this power and the corresponding efficiencies of the three stations investigated.

Table 5

Distance *d* km.	Mean intensity volt/metre	Power radiated watts	Input power watts	Efficiency
3LO 7·43	0·066	2680	5000	53·6 %
3AR 45[1]	0·0104	2430	5000	48·6 %
7ZL 20·8[1]	0·010	483	800	60·4 %

[1] This is the mean radius of the 10 mv./m. contour for these two stations, and a small correction for attenuation, based on a conductivity of 4×10^{-13} e.m.u., has been applied to 3AR.

§ 10. DISCUSSION OF STANDARDS OF SIGNAL INTENSITY FOR AUSTRALIAN CONDITIONS

English and American authorities mention the value of 10 mv./m. of intensity as constituting a continuously reliable service. Also that reception becomes unsatisfactory below 2·5 mv./m. From information supplied by local people in the areas investigated during the present work, and from personal observation wherever possible, it appears that the service limits quoted above are unnecessarily high for local conditions. Crystal sets were found in use down to 2 mv./m., while satisfactory valve reception was possible below 1 mv./m. Below 2 mv./m., however, considerable variations at night were noted, although not sufficiently large to be classified as "fading" by the local listeners. The signal never became so small as to disappear entirely.

The results of a series of transmission tests held by 3LO and 300 amateur listeners in Victoria, in February, 1928, showed that (i) fading, although not serious, was observed in many districts within 100 miles of Melbourne. The intensity at most of these places has since been found to lie between 1·5 and 0·5 mv./m. (ii) Within this area, atmospherics did not cause serious interference. (iii) Reception was unsatisfactory, in general, at places over 100 miles from Melbourne, except in Gippsland, to the east, where reception was only satisfactory up to 60 or 70 miles. The intensity at these distances is seen from the map to be less than 0·5 mv./m.

We conclude therefore that the area of satisfactory reception extends to the 2·5 mv./m. contour and that no serious interference is present down to 1 mv./m. No measurements of the "noise level" or of the prevalence of atmospherics have yet been made in the south-eastern part of Australia, but the facts quoted above indicate that both these forms of interference are probably less prevalent here than in England or America.

§ 11. ACKNOWLEDGMENTS

The greater part of the cost of this investigation was defrayed by a grant made to the University of Melbourne by the Broadcasting Company of Australia, and some of the equipment was made available by the Radio Research Board of Australia. In conclusion, I wish to express my thanks to Professor T. H. Laby for his advice during the whole course of this work, and to the staff of the Research Section of the Postmaster-General's Department for their co-operation in some of the tests.

DISCUSSION

Mr R. A. Watson Watt (who read the paper in the absence of the author). I have an uneasy feeling that the author has made unjustifiably heavy concessions to portability in his apparatus. In aiming at a high step-up $\omega L/R$ for the field loop he has been driven to working the loop dangerously near its natural frequency, and to using a tuning capacity less than three times that of the voltmeter. I would suggest that this capacity ratio has an additional disadvantage in vitiating the attempt to eliminate antenna effect. The addition of 30 $\mu\mu$f across one half of a loop tuned by a capacity of the order of 85 $\mu\mu$f should have been compensated by an equal capacity across the unused half. The discrepancy between half-loop and whole-loop e.m.f.'s (p. 197) is probably due in part to this omission. It would, too, have been desirable to include provision for measurement of loop resistance in the field, a provision which has been found necessary in most measuring sets. The very wide departure of the E/V ratios from the square of the wave-length ratio is disturbing in this connexion, for $107/86 = 1·24$ while $(484/371)^2 = 1·70$.

As regards the voltmeter, the unorthodox use of the valve cap and socket as a grid leak leads to a fear that the need for calibration at each measurement may have been mainly due to the variations of this composite resistance of large surface. Further information on the gain in sensitivity as compared with a well-insulating socket and cap and a leak of, say, 10 megohms, would be of interest, as would also information on the mode of action of such a rectifier. Mr W. G. Hayman, of the University of Western Australia, who is at present attached to the Radio Research Station, Slough, has been good enough to make for me some tests on a voltmeter set up in accordance with the author's figures. With a P.M.1.H.F. valve taken at random from stock, an arbitrarily chosen signal of frequency 1 megacycle per second gave a decrease from 0·7 ma. to 0·3 ma. in anode current with the author's free grid arrangement. The grid pin was then isolated by the cutting of the valve cap while the socket was left unaltered, and the anode current change for the same applied signal was now from 0·6 ma. to 0·06 ma. With a 5 megohm leak across the grid condenser the current changed from 0·56 ma. to 0·065 ma. on application of the signal. It would thus appear that it is more important that the grid leak proper should not be swamped by casual leakage than that it should be superseded by such leakage.

The grid current in the case last described was certainly under 0·02 μa. The mechanism of the moderately effective rectification obtained under these conditions, with i_a/e_g and i_g/e_g characteristics as in the accompanying diagram, is not clear. In any case it is difficult to see how a grid-bend rectifier can absorb less power than an anode-bend rectifier properly designed for mitigation of Miller effect; that this is a grid-bend rectifier is shown by the *decrease* of anode current on application of signal.

The phrase "field too small for satisfactory reception" (p. 198) should, of course, be qualified by the insertion of "domestic" before "reception".

It might with advantage be pointed out that the relative attenuations on two wave-lengths shown in Table 3 are roughly consistent with Sommerfeld's analysis and with Barfield's conception of a "pseudo-ground-conductivity" σ, which includes absorption by trees. The ratio of attenuation on the two wave-lengths over the same terrain is, in each case, close to the ratio of the squares of the wave-lengths. Sommerfeld's "numerical distance" is, to a crude approximation, and for considerable distances, inversely as the square of the wave-length. This relation does not, however, hold in Table 4, which might well be expanded from its present somewhat obscure form. § 9 might be amplified to show that a specialised definition of efficiency is implied in the assumption that the polar diagram of radiation is a torus of zero aperture.

I do not think that the author's conclusion as to relative noise levels in Australia and in other places can be supported by the combination of psychological and physical evidence which he adduces. It is probable that the Australian listener is more tolerant in respect of noise level than is the hypothetical listener portrayed in the very stringent specifications of service field strengths used in Great Britain and America.

These criticisms are, however, of but slight importance as against the great value of having such measurements, in another hemisphere, put before us in a form which shows very satisfactory agreement with measurements in Europe. As a worker for the Radio Research Board in Great Britain, I appreciate the privilege which I enjoy to-day of deputising for a colleague working for the Radio Research Board of Australia.

Mr R. L. SMITH-ROSE. Mr Cherry's paper is interesting, both from the practical point of view in providing data which are useful to engineers responsible for the design of broadcast transmitting stations, and from its more scientific aspect in providing additional knowledge of the electrical constants of the earth at radio-frequencies. The method adopted for making the field intensity measurements appears to depend, among other things, upon an accurate knowledge of the radio-frequency resistance of the receiving loop. It is perhaps doubtful whether this resistance can be measured with sufficient accuracy to warrant the statement on p. 198 that "the absolute value of the intensity is considered to be accurate to 1 %". This criticism does not, however, in any way influence the value of the results recorded in the paper, since a much lower order of accuracy would be adequate to illustrate the results and permit of the various deductions drawn therefrom.

I note that the conductivity obtained from the measurements made for open ground conditions in Australia is of the same order as that obtained by Mr Barfield and myself in some measurements carried out in the south of England a few years ago, by an entirely different method*. This value of the conductivity is decreased when the ground is covered by trees, as was shown by Barfield's later work which is referred to by the author. The much lower conductivity values obtained by Mr Cherry are undoubtedly due to the greater density of trees prevailing in certain parts of Australia. Similar results have been obtained by Lemoine and Rolf in the course of their investigations of the field strength from broadcasting in Sweden.

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* *Proc. R. S.* **107**, 587 (1925).

With reference to Fig. 5 of the paper, it may be pointed out that there appears to be a definite undulation in the course of the results there plotted. This undulation appears to be sufficiently regular to warrant the dismissal of the idea that it is due to unsystematic experimental errors. No reference to this fluctuation is made in Mr Cherry's paper, although I notice that in one of his previous publications† he has drawn attention to this characteristic of the graphical representation of the results of field intensity measurements obtained for transmission over sea water. In the paper referred to the author suggested that the fluctuation was due to the interference of two waves, one of which was transmitted along the earth's surface, while the other was due to reflection at the lower boundary of the Heaviside layer. Since all the measurements were made in daylight, and it is known that the intensity of such a downcoming wave is practically negligible for the wave-lengths in question, it is unlikely that this will provide a true explanation. I should like to suggest as an alternative that the second wave is obtained by reflection from the coast line on one side or the other of Port Philip Bay. At a distance of about 30 miles from the transmitting station, the ship on which the measurements were being made would be somewhere in the neighbourhood of St Leonards, and it is just possible that at this point direct and reflected waves would be obtained with a path difference sufficient to provide the ripple having a "wave-length" of about 5 miles, as indicated in Fig. 5 of the author's paper. At a distance of 50 miles the ship would be well outside Port Philip Bay, and it is likely that the distance between the points at which the two waves come into phase would be greater. This is consistent with the results recorded in Fig. 5, since the distance between the peaks increases to 11 and 16 miles as the distance of transmission is increased from 50 to 80 miles. The possibility of the reflection of waves at a coastal boundary has been pointed out previously in connexion with some experiments on wireless direction-finding carried out in this country*, while confirmatory evidence has also been provided from the results of some experiments in Palestine†.

It is a well-known occurrence in wireless direction-finding that when the path of transmission lies nearly parallel to the coast line, a deviation of the waves may take place of the order of 3° or 4° on wave-lengths in the neighbourhood of 450 metres. Where such refraction effects are observed it is reasonable to suppose that reflection may also take place. The effective coefficient of reflection will be small for the usual values of the conductivity of land and sea water, but this is consistent with the small amplitude of the ripple shown in Fig. 5 of the paper.

If such experimental evidence of coastal reflection could be confirmed by the carrying out of further experiments, a very useful purpose will have been served, since at the present time there is some discrepancy between the experimental data on coastal deviation and the theoretical explanation thereof. Perhaps the author could supply some further information or data obtained from his experiments which would assist in the further elucidation of this point.

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* *The Daylight Transmission of Wireless Waves over Sea Water* (Broadcasting Company of Australia, 1929).

† *Nature*, **116**, 426 and 498 (1925).

‡ *Nature*, **121**, 35 (1928).

Dr D. Owen said that the last speaker's suggestion was an extremely interesting one, but the effect of coast-line reflection, if real, appeared to be of small magnitude. The fluctuations of intensity shown in Fig. 5 formed only a small percentage of the total intensity.

The President said that wireless engineers would be very glad that this work was being done in Australia. Designers would be helped by the knowledge that the effects on transmission of soils and land surfaces are of the same order in that continent as in England.

The author's reply will be published in a subsequent part of the *Proceedings*.

A METHOD OF EXAMINING STEREOSCOPIC PHOTOGRAPHS

By J. M. NUTTALL, D.Sc. and E. J. WILLIAMS, Ph.D.

Communicated by Prof. W. L. Bragg, October 22, 1929.
Read by Mr T. Smith and discussed January 24, 1930.

ABSTRACT. The paper describes a method of analysing stereoscopic photographs which has been found suitable for the examination of Wilson cloud tracks. The method consists essentially in replacing the photographic plates in the cameras, illuminating them and tracing out the common image which coincides in space with the object originally photographed. A system of movable pin points is used to trace out the contour of the image, the adjustments of the cameras being kept exactly the same as when the photographs were taken. The chief advantage of the method is its simplicity—practically no computation is required and it is not necessary to know the stereoscopic angle, the magnification, etc. The method has been used by the authors to measure the ranges of secondary β-rays and in investigations on the initial directions of emission of photoelectrons, and it has been found convenient and accurate.

§ 1. INTRODUCTION

In papers on "the ranges of secondary β-rays"* and "the spatial distribution of photoelectrons"† and in a letter to *Nature*‡, we have briefly referred to our method of examining stereoscopic photographs. The method was developed originally as the result of a suggestion by Professor C. T. R. Wilson, F.R.S., and is suitable for the examination of Wilson cloud tracks.

Calculations from direct measurements of the photographs have been found to give satisfactory results in the case of α-rays, where the tracks are approximately straight, as is shown by the work of Blackett. In the case of relatively slow β-rays, as, for example, those produced by X-rays, it is well known that the tracks have generally a zig-zag appearance, and calculations from such photographs would be both difficult and tedious. Moreover, in photographing β-ray tracks many observers use two cameras with their axes inclined at a small angle, instead of at right angles, and the difficulties of calculation in such cases are enhanced. Loughridge§ has described a special apparatus, the "stereocomparator," designed for examining such stereoscopic photographs.

During the last few years we have taken a large number of photographs of β-ray tracks. The method we have developed for examining them, which will be described in this paper, has proved both accurate and convenient.

* J. M. Nuttall and E. J. Williams, *Phil. Mag.* **2**, 1110 (1926).
† E. J. Williams, J. M. Nuttall and H. S. Barlow, *Proc. R. S.* A, **121**, 612 (1928).
‡ J. M. Nuttall and E. J. Williams, *Nature*, **123**, 799 (1929).
§ D. H. Loughridge, *Phys. Rev.* **30**, 488 (1927).

§ 2. THE PRINCIPLE OF THE METHOD

Fig. 1 is a plan of the two cameras drawn to illustrate the principle of the method. Suppose that we have a pair of stereoscopic photographs P_1 and P_2 from which we desire to ascertain the relative positions of points on the object photographed or the contour of the original object. The plates, after development, are replaced in the cameras which are adjusted so that they are in exactly the same condition as when the photographs were taken. The plates are illuminated by the lamps M_1, M_2 and are viewed from the right through the camera lenses L_1, L_2. Suppose I_1, I_2 are the images on the plates of a common point T on the object photographed, and let O_1, O_2 be the centres of the lenses L_1, L_2. Since the relative positions of the plates P_1, P_2 and the lenses L_1, L_2 are the same as when the photographs were taken, then the point T', which lies on both I_1O_1 and I_2O_2, has the same position in relation to the camera as the original point T of which I_1, I_2 are the photographs. Hence by ascertaining the point of intersection of I_1O_1 and I_2O_2 we can find the position of the original point T in relation to the camera. Similarly the positions relative to the camera of any number of points T_2, T_3, ... on the original object may be obtained, and hence their positions relative to one another. This is the essential principle of the method. If the images on the plates are diffuse through insufficient depth of focus, the method can still be used and will give accurate results provided the centre of the diffuse image be chosen as I.

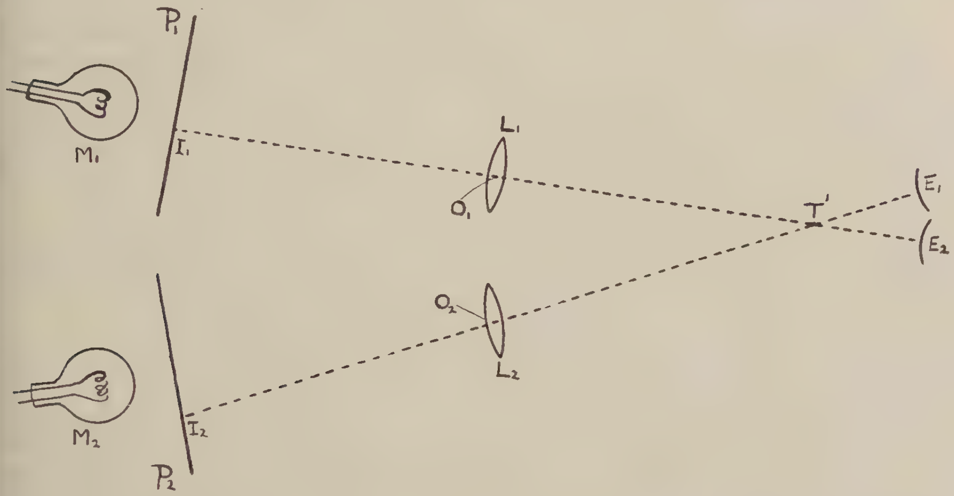

Fig. 1.

§ 3. ACTUAL PROCEDURE IN PRACTICE

An arrangement of pin points A, B, C, ..., mounted on arms with universal movements*, is placed in front of the camera (see Fig. 2) roughly in the position of the original object photographed. In order to locate and record the position relative to the camera of a point T on that object (the point T might, for instance,

* Wireless dealers supply these for use with crystal detectors.

be the origin of a Wilson cloud track), one of the pin points is moved so that it coincides with the intersection of I_1O_1 and I_2O_2, where I_1, I_2 are the photographic images of T. The fixing of a point in this way may be done by alternate adjustment of the position of the pin on I_1O_1 and I_2O_2 until it lies on *both*. In practice, however, the fixing of a pin can be done in much less time than this method of successive adjustment would require. Actually, by looking into one lens with one eye E_1 and into the other lens with the other eye E_2, an observer can readily see the photographs in stereoscopic relief, and a pin point can be brought into coincidence with the real image at the intersection of I_1O_1 and I_2O_2 as easily as it could be brought into contact with a point on a real object. The relative positions of a number of points on the original object may be recorded by the use of several pins, each pin recording the position of one point. In this way the contour of the object is mapped (for instance,

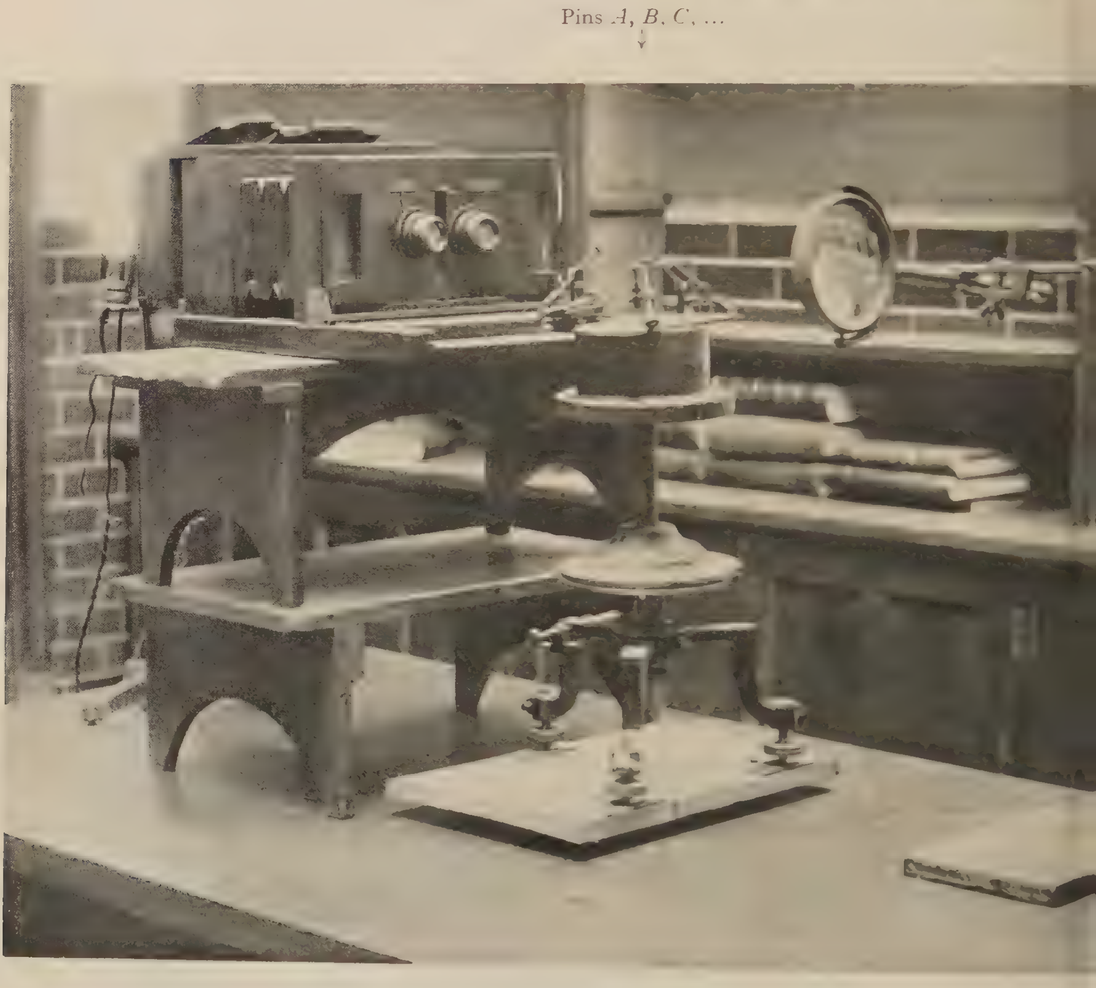

Fig. 2. Apparatus used in fixing the pins.

the contour of a Wilson cloud track), and when the pins have been "fixed" the essential process of translating the stereoscopic photographs of the original object into a spatial picture of it is completed.

Any information required about the contour of the original object may now be obtained from measurements on the pin points. Such measurements can most easily be made by projection of the images of the pins on to a screen by means of an optical lantern. In this way any desired magnification can be produced, and measurements of distances and angles which would be difficult to make on the pin points themselves become relatively easy. For this purpose it is convenient to have the pins mounted on a rotating table (a graduated spectrometer table serves admirably) so that the line joining two pin points may be brought into any desired plane relative to the direction of projection by rotation of the table*.

Owing to slight variations in the sizes of the photographic plates the fit of a plate in the holder of the camera may be slightly loose, and even though the adjustment of the camera may be the same as when the photographs were taken the plates are not necessarily in exactly the same position as at that time. This defect can easily be corrected if a reference point *R*, fixed at a known distance from the camera, is photographed at the same time as the object investigated. Then when the plates are being examined a pin point can be placed in the known position of *R* and the plates moved until the position of the point fixed from the photographs coincides with the known position. The plates are finally clamped in this position. This preliminary adjustment is not only necessary to make the reproduction accurate, but is essential if the photographs are to be seen stereoscopically at all, since stereoscopic vision is impossible if there is any relative displacement of the plates perpendicular to the plane containing the axes of the lenses. This adjustment has always to be made before proceeding with the measurements of the photographs as described previously.

§ 4. ADVANTAGES OF THE METHOD

The method which we have described in this paper has been used for the measurement of Wilson cloud tracks. The advantages of the method may be briefly summarised as follows: (1) Very little additional apparatus is required apart from the camera used in taking the photographs. (2) It is not necessary to know the stereoscopic angle, the magnification, etc., and the method involves practically no numerical or geometrical computation. Moreover, the effect of any glass apparatus between the camera and object photographed can be eliminated without calculation if such apparatus be simply replaced in the same position relative to the camera when the pins are being fixed. (3) Uncertainties due to slight variations in the stereoscopic angle, arising from the loose fit of the photographic plates in the holders or looseness of the holders in the cameras, can be eliminated. This is most important when small magnification is used. (4) In view of the fact that the images can be seen in stereoscopic relief, the process of mapping out the contour of an object takes very little time. The zig-zag appearance of the Wilson cloud track of a slow β-ray is well known, and the measurement of its total length in space from stereoscopic photographs would seem a very formidable task. Yet in our work on the ranges of secondary β-rays* the range of a β-ray was measured in about 10 minutes. (5) The accuracy of the method is limited only by the definition of the photographs. In order to test the accuracy of the method we photographed a piece of fine copper wire about 3 cm. long which was bent into a zig-zag shape somewhat resembling a Wilson cloud track of a slow β-ray. The cameras had a stereoscopic angle of about 15°, the focal length of the lenses was 15 cm., the aperture was f 5·8, and unit magnification was used. The bent wire was placed so that it pointed on the whole away from the camera, in order to test as far as possible the measurements depending on the stereoscopic effect. As the depth of focus was not great the photographs of certain parts of the wire were diffuse, so that the test was a stringent one in every respect. The developed plates were replaced in the cameras and the contour of the wire was mapped out with pin points from the stereoscopic photographs, as described above, and the total length of the wire was determined by measurement on the pin points. The original piece of wire was then straightened out and its length measured directly with a travelling microscope. The length measured indirectly from the photographs agreed within about 1 per cent. with the length measured directly with the microscope.

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* Alternatively the contour of the object could be mapped by the use of a single pin point mounted so that it could be moved by graduated screws in any one of three perpendicular directions —a travelling microscope could be adapted to give the necessary movements. One could thus obtain the three spatial co-ordinates of any point by fixing this pin point on the point of the object to be located and reading the co-ordinates of the pin from the graduated screws. This method might be specially suitable for accurate measurements on α-particle tracks.

§ 5. APPLICATIONS OF THE METHOD

Photography with two cameras at right angles is a special case of stereoscopic photography and the method is therefore still applicable in principle. But in the case of photographs taken at right angles the fixing of a pin point cannot be facilitated by stereoscopic vision, and this part of the process will therefore take a longer time.

We have used the method outlined in this paper for the measurement of the ranges of secondary β-rays, the initial directions of emission of photoelectrons, and the nuclear deflections of β-rays. In all cases we have found the method convenient in practice and accurate.

§ 6. ACKNOWLEDGMENT

Finally we wish to thank Prof. C. T. R. Wilson, F.R.S., who originally suggested we should try the method, and Prof. W. L. Bragg, F.R.S., for his help and encouragement in the various researches involving its application.

Note added January 22, 1230

Further checks on the accuracy of the method have been made by one of us in collaboration with Mr F. R. Terroux at the Cavendish Laboratory, Cambridge, and an account of the results was given in a recent paper by these authors†.

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* *Loc. cit.*

† E. J. Williams and F. R. Terroux, "The Passage of fast β-particles through Gases," *Proc. R. S.* **126**, 294, 295 (1930).

DISCUSSION

Mr T. Smith said that it was important that the lenses employed in the method should be free from imperfections, particularly coma, and that the centres of the pupils of the eyes of the observer should be placed and kept in coincidence with the projected centres of the lens apertures. Parallax methods of focussing, though almost universally recommended in text books, are carefully avoided in optical laboratories. A somewhat similar method has been used in air surveying*.

Dr L. C. Martin. It is quite true, as remarked by Mr Smith, that problems of this kind have been very thoroughly and successfully solved in connexion with air surveying, particularly in the "stereoplanigraph," an account of which has been published in the *Zeitschrift für Instrumentenkunde*. In this apparatus two perspective projections represented by photographs are mounted in "cameras" in positions related to the camera positions in the original exposures. The instrument then allows of the presentation of a stereoscopic view to the user and with the aid of a "wandering mark" a contour map of the country can then be semi-automatically drawn. Parallax observations are sometimes resorted to by persons who find a difficulty in holding the stereoscopic view; they will open one eye and then the other to see if there is any apparent displacement of the mark with respect to the image, but this is hardly likely to be a very accurate procedure.

Authors' reply. Imperfections in the lenses are no more detrimental to our method of *examining* the photographs than they are to the *production* of accurate photographs, and the error arising from this source is of the same order of magnitude whichever method is employed for the examination of the photographs. In fixing a pin the observer uses the central rays through the lenses by adjusting the position of his eye so that the image is in the middle of the field of view. Errors introduced by the use of rays through the periphery of the lenses depend on the ratio of their apertures to their distance apart in the stereoscopic camera. As the diffuseness of the photographic images of point objects is due to the rays coming from different parts of the lenses an error of the same magnitude will be incurred if, in determining the original position in space of the points photographed, measurements are not made on the centres of their diffused images on the plates. The extent to which the conditions for accurate measurement by the present method can be adhered to is shown by the very satisfactory results quoted and referred to in the paper. The method used is in certain respects the same in principle as those used in air survey and it may be regarded as an extension or perhaps a simplification of such methods. In it a full-scale reproduction of the shape of the original object is made, and no optical apparatus in addition to the camera itself is used.

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* J. C. D'Almeida, "Nouvel appareil stéréoscopique," *Comptes Rendus de l'Acad. Franç.* **47**, 61 (1858). T. Scheimpflug, "The use of the skioptikon for the production of maps and plans from photographs." *Photographische Korrespondenz*, **35**, 114 (Vienna, 1898). O. von Gruber, "The Carl Zeiss stereoplanigraph," *Zeit. für Instrumentenkunde*, **43**, 1 (1923). J. C. Kapteyn used the principle in determining the parallax of stars; several papers on this work were published about 1890.

A DETERMINATION OF THE COEFFICIENT OF DIFFUSION OF WATER VAPOUR

By W. E. SUMMERHAYS, B.Sc.

Communicated by Prof. S. W. J. Smith, December 25, 1929. *Read and discussed February* 14, 1930.

ABSTRACT. Water vapour was allowed to diffuse down a vertical tube about 2 in. in diameter, a steady concentration-gradient being maintained from approximate saturation at the top to approximate dryness at the bottom. The gradient was measured with two katharometers at a definite distance apart and the mass of vapour passing in an observed time was weighed. The result given is 0·282 cm.^{2} sec. at mean temperature 16·1° C.

§ 1. INTRODUCTION

No determination of the coefficient of diffusion of water vapour appears to have been made for thirty years, and I have been unable to find some of the original papers on the subject. In those determinations of which full particulars could be found the methods adopted seemed to be open to criticism, and it was thought that the accuracy attainable with a katharometer* used as a hygrometer would enable a method free from serious error to be found.

Two distinct methods were attempted, (1) a direct method, (2) an indirect method. Whilst the direct method gave consistent results with different experimental conditions, the indirect method had to be abandoned.

§ 2. THE DIRECT METHOD

A steady water vapour pressure-gradient was established along a vertical tube and the mass of water vapour that had passed in a known time was found. The gradient was measured by means of two katharometers inserted in the tube a known distance apart. It was considered essential that the diffusion should take place downwards as water vapour is lighter than air, so the top of the tube was covered with a damp surface and the bottom was kept dry with phosphorus pentoxide.

The apparatus consisted of a brass plate *A*, Fig. 1, 20 cm. in diameter and 1·5 cm. thick. This was placed on the base of a retort stand *R*, the shortened rod of which passed through a hole in the plate *A*: the rod thus constituted a vertical axis about which the plate could be rotated by means of a "tommy-bar" inserted in holes drilled in its edge. Two wells 6 cm. in diameter and 1 cm. deep were cut in the plate. A similar plate *B*, 1 cm. thick rested on *A*, and was prevented from rotating. In plate *B* one hole 4·85 cm. in diameter and one 6 cm. in diameter were cut. The larger hole was normally covered with a flanged brass cap *C*. Over the smaller hole was placed a copper disc E, 10 cm. in diameter and 0·6 cm. thick. This disc was perforated over an area 5 cm. in diameter with holes 2 mm. in diameter having centres 4 mm. apart. On the copper disc was placed the long tube, 4·85 cm.

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* Cf. Daynes and Shakespear, *Proc. R. S.* A, **97**, 237 (1920).

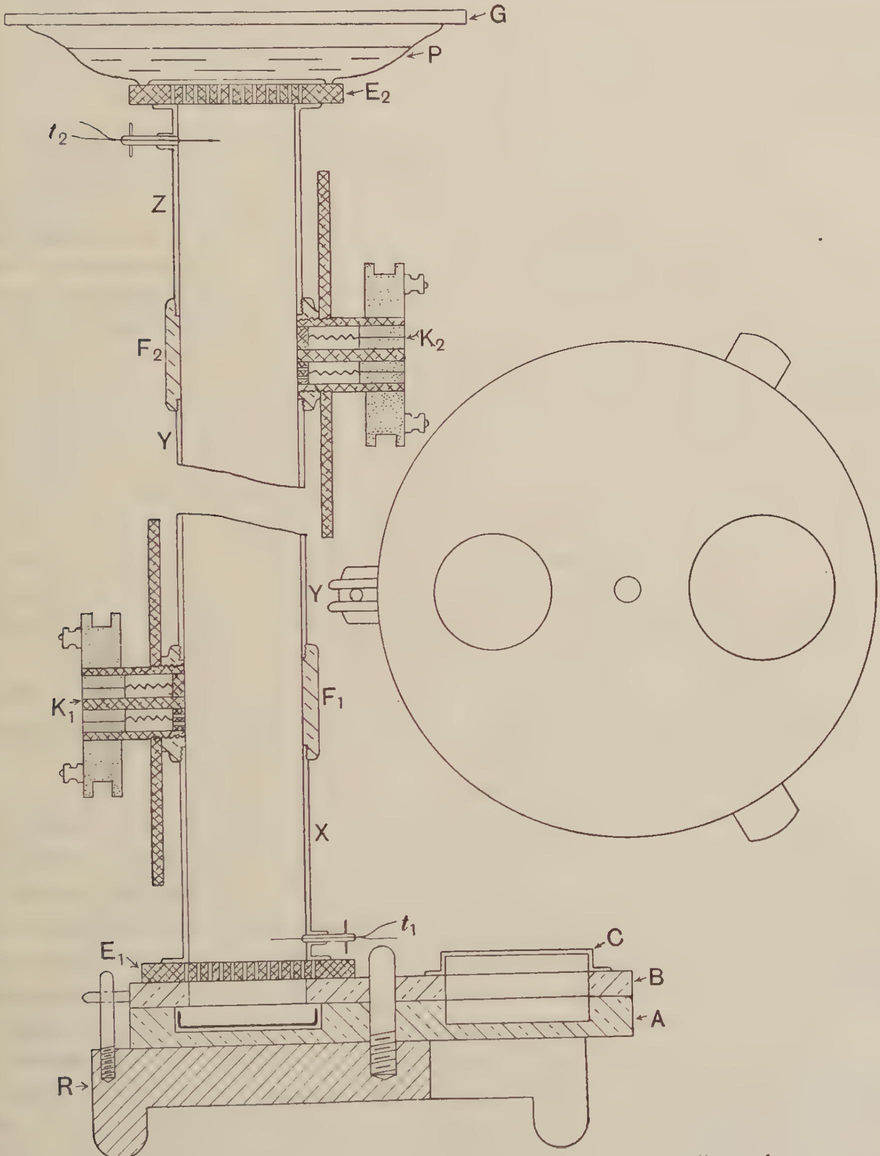

Fig. 1. Left: sectional elevation. Right: plan, with tube, perforated disc and cap removed.

in internal diameter, in which the gradient was established. This tube consisted of three pieces X, Y, Z, connected together by means of two brass collars F_1, F_2 each 4·6 cm. long into which the pieces were screwed. The top and bottom pieces of the tube were each 10 cm. long and were flanged at their free ends. The middle piece was 12·5 cm. long for half the experiments and 17·5 cm. long for the other half. Through a hole in the side of each collar a katharometer K_1, K_2 was screwed and its position adjusted until its base was flush with the inside of the collar and tube. A copper disc 15 cm. in diameter and 0·3 cm. thick (to act as a cooling fin) was threaded on each katharometer and clamped to the collar. On the top of the tube was placed another perforated copper disc E_2, but this disc had wound round its edge eight turns of manganin wire (No. 29 s.w.g.) kept in position by means of shellac. In the top and bottom pieces of the tube a short side tube was soldered 2 cm. from the flange. These tubes were normally closed with brass plugs through each of which passed one junction of a copper-constantan couple t_1t_2. During an experiment an unglazed, porous soup plate P containing distilled water was placed on the upper perforated copper disc and was covered with a glass plate G. The tube and collars were coated internally with paraffin wax and all surfaces in contact (except the top of the porous plate) were ground and greased with a mixture of beeswax and vaseline.

Preliminary experiments were made, without the use of katharometers, with a short tube of large diameter, and it was found that the cooling due to evaporation from the lower surface of the porous plate and the heating of the phosphorus pentoxide due to absorption caused the bottom of the cylinder to have a temperature higher than that at the top. A small electric heater was put in the water at the top but it was found impossible to get the temperatures equal, the temperature at the bottom being always the higher. A consideration of these facts led to the adoption of the perforated copper discs. The katharometers were designed to take 0·35 amp. and a continuous current of this value raised the temperature of the katharometer blocks in air about 2° C. above room temperature; so the katharometers were placed on opposite sides of the tube and the copper fins were fitted. When the katharometers were in place in the tube and a water vapour pressure-gradient had been established, it was found that with a continuous current the thermo-couple showed that the temperature at the top of the tube was nearly 1° C. above that at the bottom. As this temperature difference was in the right direction it was hoped that it would not give rise to convection, and a series of experiments was made with a middle piece of the tube 17·5 cm. long. Results from these experiments were moderately consistent, giving a value of 0·32 cm.^{2}/sec. for the diffusion coefficient. When the middle piece of the tube was made 12·5 cm. long and the experiments were repeated, the results were no longer consistent. The absence of convection being an essential condition for an accurate determination, it was decided to place in the cylinder along its whole length two strips of metal bent and soldered together to give a cross-shaped cross-section so as to divide the tube into four compartments. This metal was well greased and put in position and the value obtained for the coefficient was 0·28 cm.^{2}/sec., showing that convection had been present before the introduction of the metal strips. It was therefore decided to pass the current through each katharometer for 30 seconds when a reading was required, and to take readings at intervals of not less than half an hour. When this method had been adopted it was found that the temperature at the top of the tube was 0·3° C. lower than that at the bottom. By the passage of a current of 0·12 amp. through the manganin wire wound round the top copper disc *E* the temperatures at the top and bottom of the tube were sensibly equal (less than 0·1 C. difference).

To prevent the initial drop in e.m.f. that occurs when a current from accumulators is switched on, it was arranged that except during the 30 seconds when a reading was required the current should pass through a resistance approximately equal to that of either katharometer. The room chosen for the experiments was one in which temperature changes were slow, and it was found possible, by taking advantage of suitable meteorological conditions and by manipulating a very small circular window near the ceiling, to keep the room temperature constant within 0·2° C. during the whole experiment. When it was found impossible to keep the temperature within this limit the results of the experiments were rejected.

Calibration of katharometers. The katharometers were calibrated *in situ* by the pushing of air of known vapour pressure through the tube, the porous plate *P* being replaced with a ground glass plate. An aspirator of large capacity was filled with air from outside the building and was then made to push this air through a "saturator"* to the lower side-tube of the apparatus. A short rubber tube connected the upper side-tube to a "bubbler," so that the rate of flow could be observed and regulated. The saturator consisted of a copper plate 16 cm. square, on the under side of which a long spiral copper strip 1·8 cm. wide was soldered edge-on. This plate rested on the top of a large copper beaker containing water which reached half way up the spiral. Air was led into the spiral on the outside, and passed out through a tube in the centre. This central tube passed through a solid copper cylinder soldered to the top of the plate. The temperature of the air as it left the saturator was observed on a thermometer inserted in a hole drilled in the copper cylinder.

The rest of the top of the plate was covered with a thick piece of wood. An ordinary pail was lagged with felt and the copper beaker was placed on three corks on the bottom, the pail being then filled with water or ice. It was found that the temperature as shown by the thermometer could be kept constant within 0·1° C. by means of small pieces of ice dropped into the pail. A temperature about 2° C. below that of the tap water was the highest chosen. Readings of the katharometers were not considered satisfactory until they had remained constant for at least an hour and a half.

A "dry zero" for each katharometer was obtained by the passing of air from the aspirator through phosphorus pentoxide tubes and a long sulphuric acid tube. The dry zero readings are not of great value as it is found that with prolonged drying the katharometers register a gradually increasing dryness. The dry zero points shown on the calibration curves are those obtained after the passage of dry air through the apparatus for three hours, but all values of vapour pressure taken from the curves lie between points obtained with the saturator, or close to one of them. The effect on the katharometer readings of variations of atmospheric pressure were observed but were found to be very small and were sensibly equal in the two katharometers used; no correction was therefore necessary except in the calibration.

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* This "saturator" was designed by Dr Shakespear and had been used by him in calibrating katharometers for some experiments on humidity.

To start the experiment the porous plate *P* containing water and covered with the glass plate *G* was placed on the tube and a Petri dish containing P_2O_5 was put in the well under the tube. Another dish of P_2O_5 was placed in the other well and covered with the cap *C*. When the katharometer readings showed that the gradient was steady (after about two hours) another Petri dish of P_2O_5 was prepared and covered with a greased brass plate of the same diameter, and was carefully weighed. The plate adhered to the dish but could be detached by pushing it with a fine wire

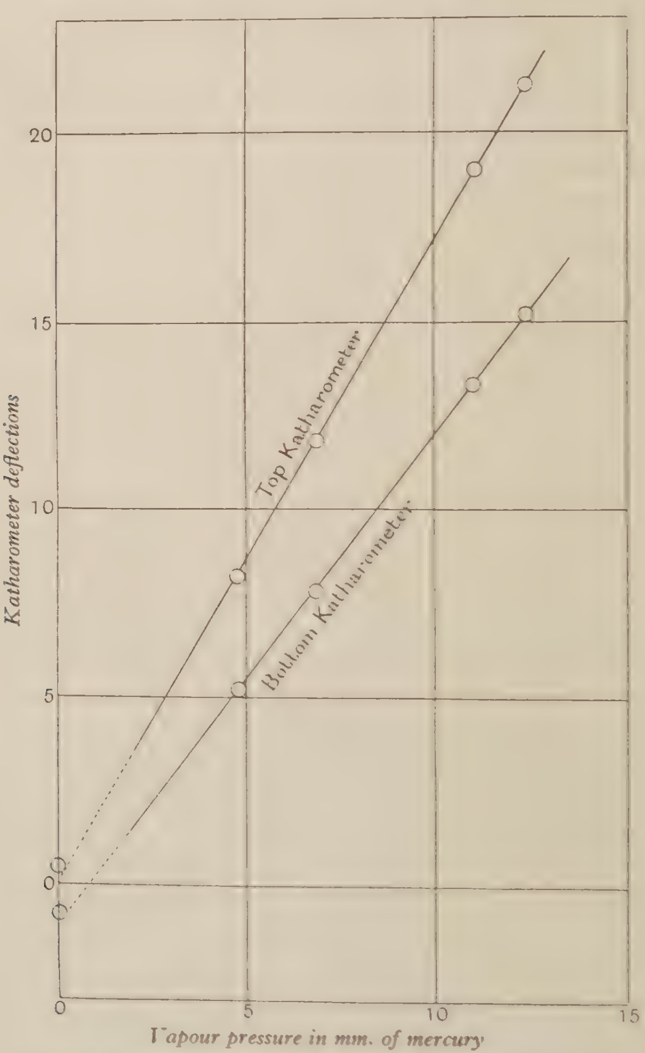

Fig. 2. Katharometer calibration curves for water vapour.

hrough a small hole drilled close to its edge. The dish of P_2O_5 in the well under the ap *C* was removed and the weighed dish substituted and its cover detached. The ap *C* was replaced and the lower plate *A* rotated until the weighed dish of P_2O_5 ame under the tube.

Readings of the katharometers were taken at intervals of about half an hour and the experiment was allowed to run for from about 3 to 5½ hours. The dish was then removed and reweighed and the diffusion coefficient calculated.

Table of Results*

Distance between katharometers (cm.)	15·2	15·2	15·2	15·2	20·0	20·0	20·0	20·0
Temperature centigrade	15·4	15·5	16·1	16·3	15·7	16·0	16·8	17·3
Bottom katharometer deflection	5·0	5·0	5·3	5·4	4·3	4·4	4·8	4·9
Vapour pressure at bottom katharometer in mm. of mercury	4·7	4·7	4·9	5·0	4·1	4·2	4·5	4·6
Top katharometer deflection	16·4	16·5	17·3	17·4	17·3	17·8	18·7	19·0
Vapour pressure at top katharometer in mm. of mercury	9·5	9·6	10·05	10·1	10·05	10·35	10·85	11·0
Difference in vapour pressures	4·8	4·9	5·15	5·1	5·95	6·15	6·35	6·4
Mass of water vapour in m.gm.	21·3	26·2	20·2	20·0	21·6	16·8	32·5	25·4
Time of experiment in min.	221	260	193	194	230	170	330	252
Coefficient of diffusion in cm.^{2}/sec.	0·274	0·281	0·282	0·278	0·284	0·289	0·280	0·285

Mean value of coefficient for 15·2 cm. distance:	0·279.
Mean value of coefficient for 20·0 cm. distance:	0·284$_5$.
Mean value of coefficient for all experiments :	0·282.
Mean temperature for all experiments:	16·1° C.

The experiments were made at room temperature and the results are for temperatures between 15·4° C. and 17·3° C. Four results were obtained with a distance of 15·2 cm. between the katharometers and four with a distance of 20·0 cm. They depend on the difference in vapour pressures obtained from the calibration curves, and the error possible here is thought to be not greater than 2 per cent.

An attempt was made to weigh the water evaporated as well as that absorbed. For this purpose a small unglazed porcelain disc was cemented near the bottom of a short brass cylinder 5·5 cm. in diameter. Water was placed in the cylinder which was then covered with a greased ground glass plate. The cylinder was placed on another greased plate whilst being weighed and was then placed on the tube after the porous plate had been removed. It was found that the loss in weight was sometimes greater and sometimes less than the gain, though the difference was always very small and could be accounted for by variations in the amount of grease adhering to the lower surface of the cylinder. The possible disturbances set up in the tube by the changing of the water vessels led to the abandonment of further work in this direction.

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* The mass of 1 litre of water vapour at n.t.p. was taken as 0·806 gm.

For steady diffusion the coefficient K is given by the equation

$$K = \frac{\text{mass}}{\text{area} \times \text{time} \times \text{concentration gradient}}.$$

For the particular application considered here this equation becomes

$$K = \frac{\text{mass} \times \text{length} \times 760 \times 1000 \times (273 + t)}{\pi \times \text{radius}^2 \times \text{time} \times 0{\cdot}806 \times 273 \times \text{vapour pressure difference}}$$

where t is the centigrade temperature.

§ 3. THE INDIRECT METHOD

The close analogy that exists between thermal conductivity and diffusion suggested the possibility of finding the diffusion coefficient by the use of a mathematical equation previously worked out for conductivity.

It has been shown that if a semi-infinite solid be at a given temperature and its bounding plane be then kept at zero temperature, the time t taken for the temperature at a depth x to reach its half value is given by the equation

$$x/2\sqrt{kt} = 0{\cdot}477^*$$

where k is the coefficient of thermal conductivity. An attempt was made to use this equation to find the diffusion coefficient.

With slight alterations the same apparatus as that used in the preceding method was adopted. The porous plate and perforated copper discs were removed and a tube 52 cm. long flanged at both ends was placed on the top of the original tube and was closed with a glass plate. The bottom piece X of the tube was replaced by a similar piece 20 cm. long. A Petri dish containing water was placed under the tube and left until the bottom katharometer K_1 gave a reading rather less than that corresponding to saturation for the temperature of the room. The dish of water was then removed and the apparatus left until both katharometers gave readings that did not alter for one hour, thus indicating uniformity of vapour pressure throughout the tube. A dish of P_2O_5 was then placed under the tube and the time taken for the bottom katharometer to reach its half value was observed. The experiment was first done before the inside of the tube was coated with wax. The values obtained in this case varied between 0·18 and 0·22. After the coating of the tube with paraffin wax the values were much more consistent with a mean of about 0·23. It appeared that the method was invalidated by the adsorption of water vapour on the inside surface of the tube and also by the unavoidable lag in katharometer indications. It was therefore abandoned.

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* Cf. Carslaw, *Introduction to the Mathematical Theory of the Conduction of Heat in Solids*, p. 34 (1921).

§ 4. ACKNOWLEDGMENTS

I have to thank Prof. S. W. J. Smith for giving me the opportunity of doing these experiments. I am very grateful to Dr G. A. Shakespear for suggesting the work, for much help and advice in carrying it out and for the loan of the katharometers and the saturator. I am also much indebted to the kindness of Messrs George Wade and Son of Burslem who supplied the porcelain discs, specially made for the purpose, free of charge; and to Mr W. Holland, of the physics workshop, for his skill in making much of the apparatus.

DISCUSSION

Mr J. H. Awbery suggested that a third katharometer, intermediate between the two actually used, might have been introduced for the purpose of ascertaining whether the concentration gradient was constant throughout the length of the tube.

Mr R. S. Whipple congratulated the author on the thoroughness of his work and suggested that temperature changes, which are the chief source of difficulty in this kind of investigation, might have been overcome by immersion of the whole apparatus in a bath of constant temperature—a method adopted by the Bureau of Standards.

Mr Rollo Appleyard. With paraffin wax directly exposed to a damp atmosphere—as was found in the tropics with early forms of electrical condensers—there is a falling off in dielectric resistance, indicating the presence of moisture. I suggest that a material better suited to the purposes of the author could be found as a lining to the tube.

Dr D. Owen. I feel some surprise at the observations of the last speaker. Experiments I have made have shown the small screening effect of an electric field due to the interposition of thin dried and waxed sheets of paper, and confirm the usual view as to the extraordinarily high insulating properties of paraffin wax, even in a moist atmosphere.

Author's reply. The nature of the problem and the design of the apparatus made the adoption of a bath of constant temperature a matter of very great difficulty. If it had not been found possible to keep the room temperature within 0·2° C. some form of jacketing would have been attempted. In the direct method with a steady water vapour pressure-gradient adsorption of water vapour on the walls of the tube should not affect the results, as a state of equilibrium would be established.

A NEW TYPE OF DEWAR FLASK, FOR USE AS A CALORIMETER

By W. L. WATTON, B.Sc., A.R.C.S., D.I.C.
Imperial College of Science and Technology

Communicated by Dr H. R. Lang, January 11, 1930.
Read and discussed, February 28, 1930.

ABSTRACT. The water equivalent of a new type of Dewar flask, of which the inside was of copper, has been measured at laboratory temperatures, and found to be more constant that the usual type. The precautions necessary in using such a calorimeter are fully discussed.

§ 1. INTRODUCTION

This series of experiments was carried out to determine the order of accuracy to be expected from calorimetric experiments which were to be made with a Dewar flask and an electric heating coil. As previous observers had found considerable variation of the water equivalent of such an arrangement*, experiments were made upon a vacuum flask in which the inside glass vessel was replaced by a similar vessel of copper.

§ 2. PRELIMINARY EXPERIMENTS

Some preliminary experiments were made by Dr H. R. Lang in June 1928 with a vessel similar to that used in the present experiments. A stirrer which had a heating coil wound on it was caused to oscillate in a vertical direction. A charcoal pocket was introduced to keep the vacuum high, and it was found advisable to use the same mass of liquid in successive experiments; a correction was made for the heat due to stirring. It appeared that the water equivalent was related to the amount of electrical power supplied. The writer set out to investigate this relation more completely, with the ultimate intention indicated above. The temperature/time results obtained in these early experiments have been plotted (§ 7) and found unsatisfactory. In the same way, an attempt to formulate the thermal effect of stirring was not very successful, the values of the relevant constant A (§ 8) differing by 50 per cent. The apparent relationship between the water equivalent and the power input is due to (*a*) insufficient stirring and inaccuracy in the evaluation of the stirring correction, and (*b*) the loss of heat due to evaporation on the wetted surface of the oscillating stirrer.

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* Experiments made by Dr H. R. Lang and Mr W. Goldsmith on an ordinary glass Dewar flask of about 800 c.c. capacity showed a sporadic variation of water equivalent of more than 20 per cent.

§ 3. THEORY AND OUTLINE OF METHOD

A heating coil is immersed in a liquid contained in the flask, along with a thermometer and a stirrer. The heat communicated to the liquid and calorimeter is evenly distributed by the stirrer, and the thermometer registers the increase of temperature of the calorimeter and its contents.

Elementary theory shows that if

K be the mechanical work done by stirring per sec. in joules,
M the mass of liquid in gm.,
s its specific heat in cal. per gm. deg. C.,
w the water equivalent of the calorimeter in cal. per deg. C.,
$(dT/dt)_m$ the rate of rise of temperature due to stirring only,
$(dT/dt)_e$ the rate of rise of temperature due to stirring and electric heating,
E the potential difference across the heating coil in volts,
C the current through the heating coil in amperes,
θ the observed temperature in degrees C. of the water bath,
T the observed temperature in degrees C. of the calorimeter,
t the time in seconds,
J the mechanical equivalent of heat in joules per calorie,
ρ_1 the cooling in deg. C. per sec. per deg. C. of temperature gradient during a stirring experiment,
ρ_2 the cooling during an experiment with current flowing,

then

$$K/J = (Ms + w)\{(dT/dt)_m + \rho_1(T - \theta)\} \qquad \text{......(1).}$$

This can be written

$$K/J = (Ms + w)/D, \text{ where } D = \{(dT/dt)_m + \rho_1(T - \theta)\}^{-1} \qquad \text{......(2).}$$

When electrical energy is supplied to the coil, this becomes

$$(K + EC)/J = (Ms + w)\{(dT/dt)_e + \rho_2(T - \theta)\} \qquad \text{......(3),}$$

which can also be expressed thus:

$$(K + EC)/J = (Ms + w)/F, \text{ where } F = \{(dT/dt)_e + \rho_2(T - \theta)\}^{-1} \text{...(4).}$$

Therefore

$$EC/J = (Ms + w)(1/F - 1/D) \qquad \text{......(5).}$$

Now if, during an experiment, $(T - \theta)$ passes from a negative value, through zero, to a positive value, the value of F (or D in the case of stirring) can be found when $T = \theta$. Hence errors due to heat loss should vanish. Even were this not so, if ρ_1 and ρ_2 are very small and almost, if not quite, equal, the error due to the neglect of the difference between the effects in the two experiments is negligible. Therefore from (5)

$$EC/J = (Ms + w)\{(dT/dt)_e - (dT/dt)_m\} \qquad \text{......(6).}$$

§ 4. MEASUREMENTS

(*a*) *Electrical measurements.* The heating current was generated by 24 Chloride batteries, each of 40 ampere-hours' capacity, connected in two banks of twelve cells, and these two banks were then connected in parallel. In series with the battery was a variable manganin resistance, a standard resistance, an ammeter and the heating coil. A manganin dummy coil of the same resistance as the heating coil could be substituted for it by a two-way mercury contact. The current was passed through this dummy coil immediately before each experiment to enable it to become steady. The standard coil was of 10 s.w.g. manganin, immersed in a water-cooled oil bath. The resistance of the coil was 0·9961 ohm at 20° C. and its temperature coefficient negligibly small.

The potential differences across the standard resistance and the heating coil were measured with a calibrated 4-dial potentiometer in terms of standard Weston-cadmium cells. A slab of bakelite, in which were mercury cups, enabled the various connexions to be made rapidly, a connector being placed in three different positions for the three quantities required.

(*b*) *Thermometry.* The temperatures of the water-bath surrounding the Dewar vessel, and of the vessel and its contents, were taken with a platinum thermometer in conjunction with a Callendar-Griffiths bridge, which was carefully calibrated in terms of the largest coil. The thermometer had a fundamental interval of 4·775 ohms. The temperature of the laboratory was taken with a mercury thermometer, graduated in half-degrees; the ice and steam points were found to be at + 0·7° C. and 100·1° C. respectively, when corrected; readings of the mercury thermometer have been corrected accordingly.

(*c*) *Weight and volume.* The volume of water necessary to fill the calorimeter up to 2 cm. from the top was found by trial, and a file mark corresponding to this volume was made upon the neck of a flask. The flask was weighed with calibrated brass weights when filled to the mark and also after the contents had been emptied out. The volume of water used was 559·2 cc. at 17° C. It was found that, at the same temperature, the same mass of water could be measured off by volume to 0·2 gm. In most cases, the water was measured off in this way.

(*d*) *Time.* The rate of rise was measured as described below (§ 6). The standard clock marking seconds on the chronograph was rated against the broadcast Greenwich time signals. It was found to have a variable rate, from zero to a loss of 20 seconds a day. This error, though large for such a clock, would still not be sufficient to affect the results. Other times were taken with a ship's chronometer.

§ 5. THE CALORIMETER

The Dewar flask was about 50 cm. long and 5 cm. in diameter, and had a capacity of about 800 cc. The usual inside of glass was replaced by a copper vessel of similar shape and size, the joint being made air-tight with sealing wax. The glass was made by the National Glass Industry of London. The joint (*R*, Fig. 1) was about 1 cm. from the top of the flask and an equal distance above the level *V* of the water inside, when the apparatus was completely assembled. The exhausting tube at the bottom *A* was connected to a Hyvac pump and to a discharge tube and charcoal pocket. The space was first evacuated by the Hyvac pump, and more completely by the charcoal pocket in liquid air. In less than 15 minutes a discharge coil capable of giving a 1 cm. spark in air was unable to produce a discharge across the discharge tube. This tube was about 16 cm. long and was of the usual type.

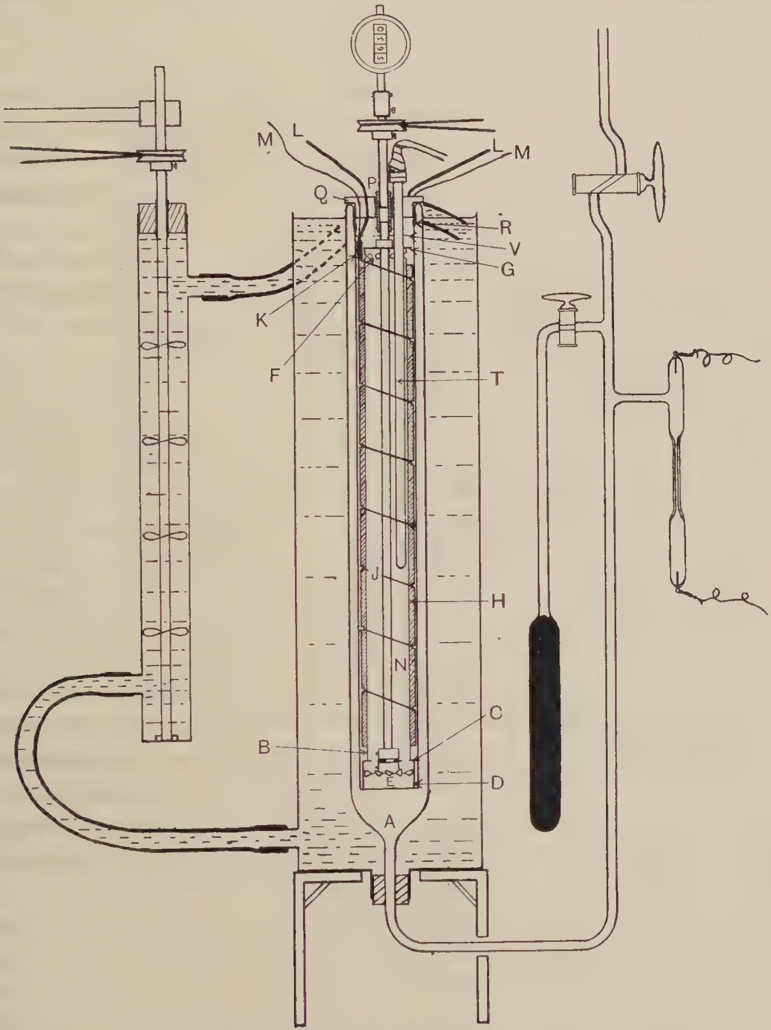

Fig. 1. Dewar flask with copper inner vessel, arranged as calorimeter.

The modified Dewar flask was almost immersed in a lagged water bath, about 2 cm. being above the surface. The water was made to circulate by means of a pump driven by an electric motor. The temperature of the jacket did not change by more than 0·5° C. per hour on any occasion, and the variation was usually less than this.

The narrow and comparatively long shape of the calorimeter placed considerable limitations upon the form of the heating and stirring arrangements. Particularly was it difficult to find a piece of apparatus of the required compass which would ensure that the water was thoroughly mixed and the heat distributed uniformly throughout the whole mass. Several months were spent before an efficient stirring system was designed, and even with this form of stirrer it was found necessary to stir at the rate of 35 revolutions a second if the results were to be consistent. A copper cylinder *B* (Fig. 1) was closed at the top and bottom with brass plates *G* and *C*. A bearing in each held the propeller shaft. The bottom disk *C* was pierced with several holes so that it was a mere grid of metal. Equally spaced around its circumference were four supports *D*, portions of a cylindrical shell, inside which the multibladed propeller *E* revolved. The top disk *G* was coarsely serrated around the circumference which was a little larger than that of the copper cylinder. In the top of this cylinder, and about 1 cm. beneath the top baffle plate *G*, were bored a large number of holes *F*. The diameter of both baffle plates was a little larger than that of the cylinder, and was such that they loosely fitted the inside of the calorimeter. The propeller *E* was revolved in such a direction as to suck water down the inside of the copper cylinder and pump it up the outside. In each circulation the water had to pass through two baffle plates and suffer churning up in the propeller chamber. Moreover, should the water inside the copper cylinder have a slightly different temperature from that surrounding it, the high conductivity of the copper of the cylinder would promote an equalisation of temperature. The thermometer *T* was placed inside the copper cylinder through holes made to fit it tightly.

Around the outside of the copper cylinder and equally distributed were six long, thin ribs *H* of bakelite; the heating coil *J* was wound non-inductively in notches in these. In order that the heating effect might be distributed over a large volume, a distance of about 2 cm. separated adjacent turns. For ease in working later, the resistance was adjusted to be nearly 1 ohm, and about $1\frac{1}{2}$ metres of 20 s.w.g. manganin wire were employed. In order to minimise errors due to a wrong position of the potential leads, the ends of this coil were slipped into a hollow copper cylinder *K* formed by boring a hole to fit the wire in pieces of 12 s.w.g. wire about 1 cm. long. The external current leads *L* were of 16 s.w.g. copper wire, the ends of which were filed down until they fitted in the open ends of the connecting pieces. The wires were soft-soldered in this position, the solder filling the inside spaces and making a solid joint. The potential leads *M* were of 28 s.w.g. silk-covered copper wire. The covering was removed from that portion of the wire which was inside the calorimeter, and the wire was soldered to the end of the connecting piece *K* nearest the heating coil. All the bare wire in the calorimeter was then coated with an alcoholic solution of shellac and a current of some 5 amperes was passed through it so as to bake the shellac. Three coats were put on in this way.

The shaft of the stirrer *N* was in two parts connected by a bakelite sleeve *P*. This served to diminish the conduction loss along the shaft.

A lid of box-wood *Q* was pierced with holes to take the four leads, the stirrer shaft and the thermometer. The holes for the current leads tightly fitted the wires, the rigidity of which kept the lid in place. The lid fitted the top of the calorimeter closely, and evaporation during an experiment must have been small. The stirrer was driven by a shunt wound motor, a controlling resistance being in series with the armature.

§ 6. METHOD OF MAKING AN EXPERIMENT

The pump circulating the water in the jacket was set going about 20 minutes before the experiment proper. The heating current was adjusted to the required value while passing through the dummy heating coil, and the measuring flask was filled with distilled water and cooled. The charcoal pocket was next immersed in liquid air, and by this time the temperature of the water bath was steady and its value could be determined, the time shown by the chronometer being noted. The temperature of the laboratory was observed immediately afterwards.

The measuring flask was removed from its cooling bath and the contents, which were about 10° below room-temperature, were emptied into the Dewar vessel. The stirring mechanism was inserted and the leads were connected to a row of four conveniently placed terminals. The platinum thermometer was inserted after drying, a pulley and collar were screwed on to the stirring shaft, and the revolution counter was fitted to the collar. The experiment was not proceeded with until the nature of the vacuum had been tested and found to be "black."

The stirrer was started and its speed adjusted by trial to the required value. At first some trouble was experienced in keeping the speed of stirring reasonably constant, until the original motor was exchanged for another of a modern type, made by the General Electric Company, of 1/10 h.p. This ran fairly constantly, the total variation at the slowest speeds being no greater than 3 per cent. and at higher speeds very much less. This variation was probably due in some measure to the fluctuating voltage of the mains from which its current was derived. The stirrer shaft revolved about three times as fast as the motor.

The current was switched over from the dummy heater, and the chronograph was set going. At some definite time, as read from the chronometer, the experimenter made a mark on the chronograph drum by tapping the key and allowing it to rise in unison with the half-second beats of the chronometer. In this way any particular record on the chronograph could be referred to events happening at the same time in other determinations.

The bridge was set to be balanced at a temperature corresponding to 15 cm. of bridge wire (about 3° C.) below the jacket temperature, and the chronograph key was depressed as the galvanometer spot passed through zero, which was set to be at the position of balance. The galvanometer had excellent zero-keeping properties and these were carefully preserved by fairly frequent reversal of the current when the deflection was at all large. Any change of zero was carefully noted at the end of each experiment; but this occurred to so small an extent as to render any error due to it very small. Such errors, unless they were progressive, which is doubted, were eliminated in the method of reducing the results. In this way, a mark was obtained for each 5 cm. of bridge wire until the calorimeter had reached a temperature corresponding to 15 cm. above the jacket temperature (as observed at the beginning of the experiment).

Meanwhile, at intervals of a minute, the p.d. across the standard resistance and across the heater were determined alternately by a second observer. The reading of the counter was taken periodically. At the end of an experiment, the vacuum was tested again, and the jacket and laboratory temperature were redetermined, the time being noted.

The water in the Dewar flask was sucked out by a filter pump. The flask was then dried with a mop, some methylated spirits were poured in and mopped out, and the inside was finally dried by the blowing in of air until it appeared dry by inspection and there was no smell of methylated spirits.

§ 7. REDUCTION OF RESULTS

After removal of the record from the chronograph the whole minutes, indicated by a special mark made by the clock, were numbered from zero. By reference to the mark indicating a known chronometer time, the chronometer time corresponding to the zero minute mark was known. This was called "zero time." Hence if the chronometer time corresponding to any other mark was required, it could be obtained instantly by adding zero time to the chronograph time of that record.

In a similar way, the marks corresponding to each 5 cm. of bridge wire were given serial numbers commencing with 1, and these numbers were denoted collectively by the symbol G. The chronograph time for each of the marks was read off to the nearest tenth of a second, a procedure which was simplified by adjusting the chronograph so that 1 cm. of record corresponded to 1 second of time. The time interval between each successive pair of marks was obtained by subtraction, and the intervals were denoted by the symbol F. (In stirring experiments, the symbol D was used because the range of temperature between two marks was 1/10 that used in the main experiments.)

The intervals F as ordinates were plotted against the corresponding mean values of G as abscissae. The points were found to lie on a straight line (referred to as a G line) as required by the approximate expansions of equations (2) or (4); for G bears a linear relationship to T. Sometimes the points were very far from being collinear; this was almost invariably found to be the case when the stirrer was rotating at a speed lower than 35 revolutions a second. In most of the experiments described, the stirring was faster than this; other experiments were taken into consideration only if the G line was satisfactory. This linear relationship seemed to be a crucial test of efficient stirring, and was so regarded in these experiments. Typical G lines for (*a*) good, and (*b*) fair, stirring conditions are shown in Fig. 2.

The mean jacket temperature, during the period when records were being made, was calculated by simple proportion, and the G value corresponding to this temperature was worked out. This value of G when corrected for lag was that at which the jacket and calorimeter temperatures were the same and was denoted by G_0. The corresponding value of F (namely F_0) was read from the G line.

But from equation (4)

$$F_0 = 1/(dT/dt)_e,$$

whence $(dT/dt)_e$ is known. Stirring experiments giving the value of $(dT/dt)_m$ were worked out in the same way. The reciprocals of F_0 were calculated and the stirring correction was subtracted. The difference was converted to platinum degrees and then to the gas scale, the mean correction over the range of temperatures used in the experiment being adopted. The electrical power supplied was calculated from the potentiometer readings on the assumption that the e.m.f. of the cells was 1·0183 volts. The change of e.m.f. with temperature over the relevant range was so small that a correction was not necessary. This fact and the value of the e.m.f. were verified experimentally by comparison with N.P.L. standards. The temperature variation of the heat-capacity of the volume of water used was allowed for, the weight being corrected for the buoyancy of the air. The mechanical equivalent of heat was taken as 4·180 joules per calorie.

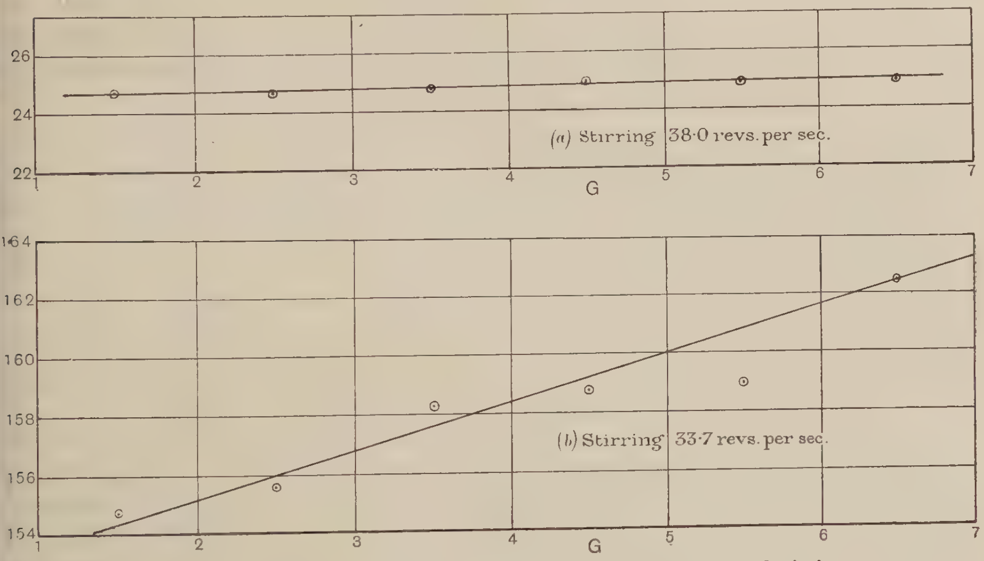

Fig. 2. (*a*) Above critical rate of stirring; (*b*) below critical rate of stirring.

§ 8. STIRRING

Experiments on the correction for stirring were carried out in the same way as the main experiments, except that the temperature interval over which the time to rise was taken was diminished to 1/10 or 1/20 of that in the main experiments. Much greater difficulty was experienced in obtaining consistent results for the rate of rise with stirring than was experienced when electrical energy was used. The rate of rise in cm. of bridge wire per second at the mean jacket temperature was derived from G lines. It was found that this rate $1/D_0$ was connected with the rate of stirring by the law

$$A = D_0 r^n;$$

for, provided that the stirring was fast enough to mix the liquid thoroughly, when $\log D_0$ was plotted against $\log r$ the points were on a straight line, from the slope of which the value of n could be obtained. It was found that, in general, when the stirring was below 35 rev. per sec. (i) a G line could not be drawn accurately, if at all, and (ii) the stirring did not obey the law quoted or any other ascertainable law, being quite inconsistent.

Above 35 rev. per sec.*, however, the law was obeyed with fair accuracy, the value of n being 2·20 while the mean value of A was $3{\cdot}03 \times 10^5$ when r was expressed in rev. per sec. and D_0 in cm. of bridge wire per sec.

Table 1 shows the accuracy obtained, the correction calculated being compared with that obtained experimentally. Owing to the difficulty of obtaining reliable values experimentally, it is considered that the calculated correction is more accurate than a single observed value and it has accordingly been applied to the experiments.

Table 1. Stirring corrections

Rate, r.p.s. ...	31·0	33·8	34·4	35·2	38·5	43·0
Observed	0·0064	0·0068	0·0079	0·0089	0·0099	0·0131
Calculated	0·0063	0·0076	0·0079	0·0083	0·0102	0·0129

The stirring correction amounted to 1/5 of the total heating effect when the electrical power was 20 watts; but diminished in importance as the power increased. Accordingly, great care was taken to obtain a sufficiently accurate value of the correction.

§ 9. SOURCES OF ERROR

Heat loss. Errors due to this cause were reduced to a minimum by the careful maintenance of a high vacuum between the vessels of the Dewar flask, and by details in the design of and method of working the apparatus already described.

Lag of the thermometer. The platinum thermometer had an appreciable lag, though encased in the thinnest glass tube available. This lead to two errors in the value of F_0 (or D_0) as read from a G line: (i) when the thermometer registered the same temperature as the jacket, the temperature of the calorimeter was higher than this: hence the condition $T = \theta$ was not satisfied; (ii) since the lag was assumed to be proportional to the rate of rise of temperature, the amount of lag would change with F and the G line would be turned through an angle.

The first effect was found to cause an error of about 1 per cent. on the value of the water equivalent, but the error due to the second cause was about 0·1 per cent. and was therefore negligible.

The thermometer was cooled and then placed in the water bath which was at room temperature, and was continually stirred. A series of readings was obtained with the galvanometer and chronograph. These were plotted on a time/temperature curve to test their regularity.

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* Occasionally satisfactory stirring conditions obtained at lower speeds than this.

Elementary theory shows that

$$dT/dt = (\Theta - T)/\lambda \qquad \text{......(7)},$$

where Θ is the temperature of the liquid,
T the temperature of the thermometer at any time t,
λ a constant (with dimensions of time).

We have also

$$\Theta = \Theta_0 + mt, \text{ where } m = d\Theta/dt.$$

By integrating equation (7) we have

$$\lambda = t/\log_e \{(T_0 - \Theta)/(T - \Theta)\} \qquad \text{......(8).}$$

The values of t and $\log_e \{(T_0 - \Theta)/(T - \Theta)\}$ were plotted and λ was found to be 15·0 sec. The value of m was obtained from the appropriate G line and hence the product $m\lambda$. This gave the lag of the thermometer in centimetres of bridge wire, and was subtracted from the values of G_0 before the corresponding F_0 values were read from the G line.

§ 10. SUMMARY OF RESULTS

The values obtained for the water equivalent are given in the seventh column of Table 2, which gives also the experimental results from which the water equivalent was calculated.

Table 2. Water equivalent of Dewar flask

1	2	3	4	5	6	7
76·6	0·01240	35·70	29·2	18·23	19·0	130
77·4	0·01223	35·15	29·4	17·96	18·4	129
24·8	0·03976	114·7	38·0	18·93	19·4	131
33·7	0·02887	83·51	37·2	18·13	18·4	133
34·2	0·02840	82·12	37·5	18·25	18·6	133
44·0	0·02121	61·05	40·7	18·44	19·6	130
57·8	0·01614	46·41	38·8	19·24	20·4	129
70·6	0·01271	36·21	37·6	18·30	20·5	123
109·5	0·00764	21·99	36·2	19·60	20·4	130
65·2	0·01431	41·15	34·0	19·10	18·7	129
87·4	0·01023	29·43	34·5	19·29	19·5	129
71·6	0·01231	35·88	39·2	20·74	21·5	138
51·9	0·01809	52·43	36·7	19·46	18·3	134
40·0	0·02395	69·42	37·5	19·09	18·8	134

Column 1 gives the value F_0 as obtained from the G curve, in sec. per cm. of bridge wire; 2, the rate of rise of temperature due to electrical energy in deg. C.; 3, the electrical power used in watts; 4, the rate of stirring in rev. per sec.; 5, the mean water bath temperature in degrees C.; 6, the mean laboratory temperature in degrees C.; and 7, the water equivalent in gm. calories per degree C.

11. COMPARISON WITH RESULTS OBTAINED BY OTHER OBSERVERS

Griffiths* in his experiments on aniline found that the stirring correction was about 1/25 of the total heating effect. When water was used, he experienced trouble due to electrolysis; but his German silver wire was uncovered and there was a considerable p.d. across his heating coil. The resistance of the heating coil in the experiments herein described was constant to 0·001 ohm.

Griffiths† experienced considerable difficulty in his experiments upon *J*. His observation that the thermometer readings were not steady and that the water equivalent was inconsistent with low-speed stirring are confirmed by the present experiments. He used a water-motor to stir his calorimeter, and was troubled by its variable speed. In a somewhat smaller degree, the author experienced this difficulty, and this fact may account for the difficulty in obtaining really good stirring corrections. In the *J* experiments the rate of stirring used was of the order of 2000 rev. per minute, and the heat due to it varied from 1/6 to 1/10 of the heat due to electrical energy only. The heat due to stirring under the same conditions varied by about 5 per cent. Griffiths attributes this to the variation in the heat generated at the bearings. This may have occurred in the present experiments also, though there is no direct evidence of it.

W. R. and W. E. Bousfield‡ used a Dewar flask of much larger capacity, about 3000 c.c. A much larger amount of electrical energy was supplied to heat this up, yet only a single blade propeller revolving at 340 rev. per minute was used. The author has found that such apparatus does little else than cause the mass of water to circulate *en bloc*. Its large diameter renders it very susceptible to loss of heat by evaporation and similar causes, and thus makes for some uncertainty in the results. Indeed, the experimental evidence obtained by both Griffiths and the author casts considerable doubt upon the efficiency of the stirring and hence upon the accuracy of the results obtained. The water equivalent in the Bousfields' experiments was 1/6 of the thermal capacity of the calorimeter and its contents, and the greatest discrepancy in the water equivalent was about 1 per cent.

§ 12. DISCUSSION OF RESULTS

The experiments herein described show how important it is to ensure that the stirring is really mixing the contents of the calorimeter thoroughly. It appears that the water equivalent is independent of any of the quantities measured, and that the mean value of a number of determinations will accordingly be the most probable value. The experiments serve to demonstrate that, provided the necessary precautions be taken, the water equivalent can be measured with an accuracy of about 1 per cent., so that calorimetric experiments should be accurate to 0·5 per cent. or, better, the accuracy depending upon what fraction the water equivalent is of the whole.

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* *Proc. Phys. Soc.* **13**, 234 (1894).

† *Phil. Trans.* **184 A**, 361 (1893).

‡ *Phil. Trans.* **211** A, 199 (1911).

§ 13. ACKNOWLEDGMENTS

The author desires to express his gratitude to Dr H. R. Lang for much helpful assistance and advice, and to Mr R. Jessel, who assisted in the observations throughout the experiments. Most of the apparatus was made in the physics workshop, and the author's thanks are due to the Superintendent, Mr W. J. Colebrook, and the members of his staff, for help in design and construction.

DISCUSSION

Mr R. S. Whipple congratulated the author on a thorough and useful piece of work. The efficiency of stirring was an important requirement which the author had successfully met; failure in this respect was a common source of error in such experiments. He agreed with the author's finding that Dewar flasks are of little use for calorimetry; further, their heat-retaining properties vary very widely from one flask to another.

Dr H. R. Lang. I wish to emphasise that the inner vessel is not copper-lined, but is itself actually made of thin copper. It was intended that the sealing wax joint *R*, Fig. 1, should eventually be replaced by a fused copper to glass joint, such as is made for large wireless valves.

In many experiments on the specific heat of oils, made of recent years in America, Dewar flasks have been used, and, despite the frequent remarks by other observers as to the vital importance of efficient stirring, insufficient attention has been paid to this matter. I would go further than Mr Whipple and not only urge greater attention to the point, but suggest that some such tests as the plotting of "G" lines should be carried out in every case so as to ensure that the stirring is satisfactory.

Mr J. H. Brinkworth. Thanks are due to the author for this work which adds very materially to our knowledge of the vagaries in behaviour exhibited by vacuum vessels. May I emphasise the importance of always having an arrangement whereby the quality of the vacuum can be continuously observed during experimental measurements. To illustrate this I show some figures relating to heat losses observed in the determination of the specific heat of steam near 100° C. by the continuous flow method. The diagram indicates that in Callendar's experiment the glass vacuum jackets always cracked. I experienced the same kind of trouble when using glass calorimeters myself, and in consequence I designed the silica flow tube. It should be noted that the effect of silvering is quite unimportant when there is no vacuum.

With more special reference to the paper under consideration, it would be very interesting to know if sporadic variations in the value of the water equivalent would be observed if experiments were made on all-glass vessels in which the perfection of the vacuum was maintained: and would the water equivalent vary with the depth of water in the inner tube?

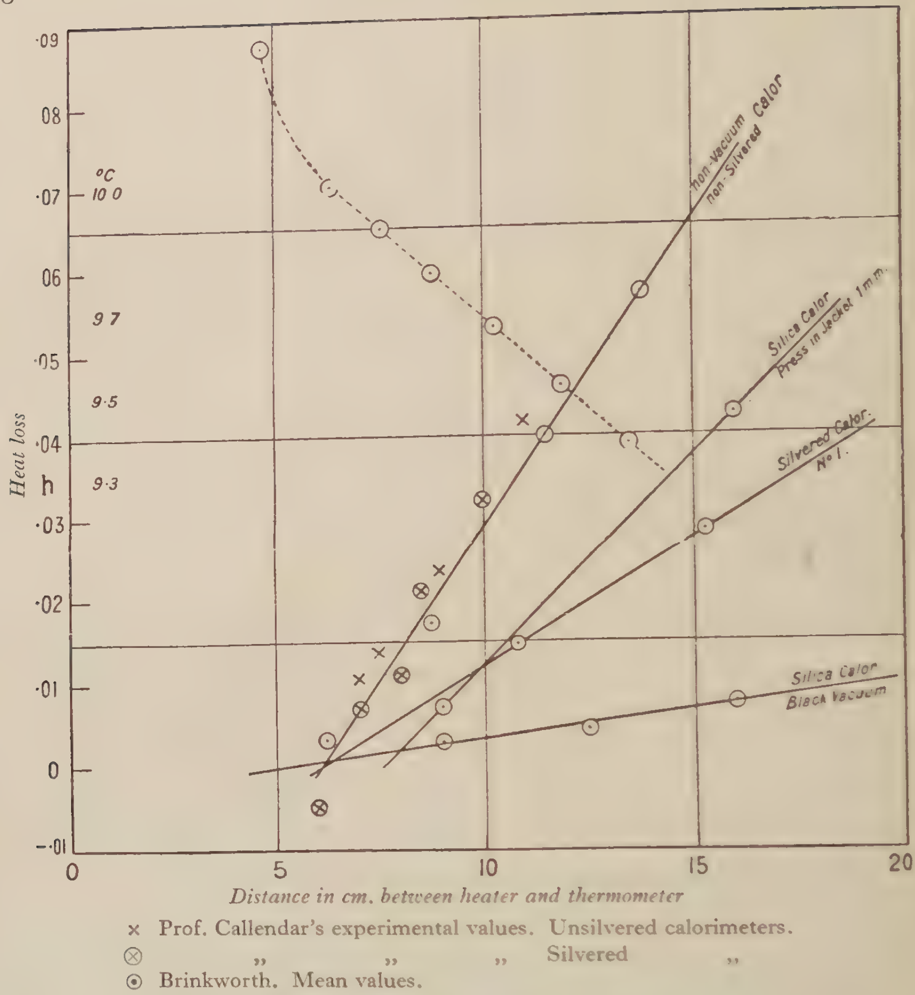

Diagram illustrating Mr Brinkworth's remarks.

AUTHOR'S REPLY. I agree with Dr Lang's remarks concerning the importance of actually testing the efficiency of stirring. Wherever possible, a quantitative test should be applied, for I have found that visual tests, such as the dropping of ink into water which is being stirred, are unreliable. It was because of Mr Brinkworth's work upon the heat loss from vacuum jackets that I maintained a high vacuum between the walls of the calorimeter. My experiments have shown the necessity for constantly testing the perfection of the vacuum. It is possible that the variation of heat-retaining properties among flasks of the same type, observed by Mr Whipple, may be due to variations in the degree of perfection of the vacuum.

I have no reliable evidence concerning the effect of the depth of water in the vessel upon the water equivalent, since this depth was fixed by the stirring apparatus. So far as I am aware, no reliable experiments have been made with an all-glass flask in which a high vacuum has been maintained.

THE MASSES OF THE PROTON AND ELECTRON

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ABSTRACT. An application of a principle proposed by the author and described as the principle of minimum proper time is made to obtain an equation for the ratio of the masses of the proton and electron. The result obtained is in agreement with the experimental value. The possibility of explaining the asymmetry in regard to the masses by means of a metrical consideration is discussed.

The new quantum theory in its later developments indicates that relations exist between certain physical quantities which have hitherto been regarded as independent. This has led to a search for these relations and we have as a notable example Eddington's attempt to express the fundamental charge e in terms of h and c. We will consider here another relation.

It has been suggested* that the world line of an electron is to be regarded as made up of fundamental units of length of magnitude h/m_0c, where m_0 is the electronic mass; similarly the world line of the proton is composed of units of length h/M_0c, and that in the study of these fundamental masses no shorter length associated with their world lines will ever be revealed.

Perhaps the most convenient expression of this theory is by means of a principle of minimum proper time which states that in association with the electron (or proton) no proper time less than h/m_0c^2 (or h/M_0c^2) will be observed. This is, of course, different from the Bohr-Heisenberg uncertainty principle which states that the position of an electron can be determined as accurately as we wish but that the momentum can then be determined only to an order of accuracy given by the equation $\Delta q \Delta p \sim h$, where Δq denotes the error in the determination of position, Δp that in momentum. It has, however, been shown that in atomic dimensions the application of the principle leads to no contradiction to the uncertainty principle.

In a recent paper Fürth† arrives at the principle by using the uncertainty relation and modifying it by assuming that the electron cannot be located as accurately as we please. By means of a number of rather arbitrary assumptions he arrives at a value of the ratio M_0/m_0 and it is the determination of this ratio that we make the subject of the present paper. We shall approach it purely in the light of the principle of minimum proper time.

Fürth states that the value h/m_0c is to be regarded as the radius of the electron, but as this value is 2000 times the accepted value he inclines to the belief that the principle can apply only to neutral masses. There is no reason to regard h/m_0c as the electronic radius nor to limit it to neutral masses. The establishment of the principle of minimum proper time shows that it is true for charged or uncharged masses and there is no suggestion that the fundamental length means more than a length along the world line. Fürth's results are, however, very interesting and we can remove his assumption.

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* *Proc. R. S.* A, **117**, 630 (1928).

† *Phys. Zeit.* **30**, 895 (1929).

Let τ_0 denote the minimum proper time h/m_0c^2. If it were possible for the mass m_0 to be converted to radiation we could presumably calculate the frequency of that radiation by means of the equation $m_0c^2 = h\nu_0$ and if the proper time is not less than τ_0 the radiation arising from this transformation of matter will have ν_0 as a maximum frequency. The proper time is the ordinary time in the system in which the electron is at rest, so that the frequency is calculated in that system.

It is not possible to contemplate this transformation in the case of an electron alone, since this would require the disappearance of charge, but we may suppose it to occur in the case of an electron and a proton. In this case we consider the two bodies as a single one of mass $(M_0 + m_0)$. The new quantum theory teaches that it is necessary to consider phenomena as possessing a double-sided character, one aspect being that in which we contemplate particles and in the other we contemplate waves; so that we can consider the problem as that of a particle of mass $(M_0 + m_0)$ and as that of radiation of a certain wave-length. The principle of minimum proper time applied to this union of electron and proton shows that the maximum frequency possible is ν_0 where

$$\nu_0 = (M_0 + m_0)\, c^2/h,$$

or the minimum wave-length is λ_0 where

$$\lambda_0 = h/(M_0 + m_0)\, c.$$

Now the wave-length in radiation is a unit corresponding to the unit along the world line of the particle. If we enquire what this unit could correspond to in the case considered, the suggestion at once arises that it must correspond to the distance between the centres of the electron and proton. Our method of approach to this point is rather different from that of Fürth but this suggestion is substantially due to him.

The older theories give as the value of the radius of the electron $r_0 = ke^2/m_0c^2$ and of the proton $R_0 = ke^2/M_0c^2$. The value of k depends upon the distribution of the charge, and the values $k = \frac{1}{2}$ and $k = \frac{2}{3}$ have been considered in the classical theory. If, however, the charge is regarded in the light of the new quantum theory it becomes difficult to consider an electron or proton as a sharply defined structure, and we can only speak of a radius as an "equivalent" one. This Fürth does, and with a rather arbitrary choice of equivalent radius he finds $k = 15/32$. We can, however, still speak of a distance between the centres of the two bodies and we may write for it the value:

$$d_0 = k\frac{e^2}{c^2}\left(\frac{1}{m_0} + \frac{1}{M_0}\right),$$

since the dimensions of the charges appear to be proportional to $e^2/c^2 \times$ mass.

We have thus:

$$\frac{h}{(M_0 + m_0)\,c} = \frac{ke^2}{c^2}\left(\frac{1}{m_0} + \frac{1}{M_0}\right)$$

or writing

$$\mu = M_0/m_0$$

$$\mu + (1/\mu) + 2 = hc/ke^2.$$

The occurrence of hc/e^2 is noteworthy. It occurs in Eddington's calculation of e and in the determination of the number of chemical elements by the application of the principle of minimum proper time; and we note that it is a pure number, having no dimensions.

We cannot proceed further to examine whether the equation gives the experimental value for μ unless we know k accurately. If we take the classical view and write $k = \frac{1}{2}$ or $\frac{2}{3}$, μ is of the right order but the value is too small. The result is, however, striking, for we have an equation in μ thus reducing the number of fundamental constants by one and the failure to obtain the correct value is explained by our ignorance of k. If we accept Fürth's assumption that the equivalent radius must be calculated on the assumption that it is the radius of the sphere within which all the charge must be enclosed so that it produces the same moment as the charge in the actual distribution assumed in the new quantum theory, $k = 15/32$ and the value obtained for μ (1838·2) is then in agreement with the experimental value (1838·3).

One of the most interesting and puzzling things in atomic physics is the asymmetry with regard to mass in the case of proton and electron. The question is, why is the positive charge associated with a mass so widely different from that associated with the negative charge? In theoretical considerations the asymmetry is very striking. In one line of attack, that of the five-dimensional theory, we meet with a constant α which may have the values $\pm e/m_0c$. Thus the theory seems to account for the occurrence of positive and negative charges, but there is no suggestion that m_0 has more than one value. It would be a great step if we could in some way relate m_0 and M_0, and this makes it worth while to consider attempts in this direction such as that given above.

We will consider the problem from another point of view, though the asymmetry is replaced by another. Nevertheless a change in the point of view may be useful. It has been suggested that atomic phenomena are indicative of a special kind of metric preferred in nature. In this we are following the way opened up by Weyl and Eddington in their inclusion of electromagnetism with gravitation into a space-metric system. We take the view that it is not in electromagnetic phenomena that the metric is to be found, but in quantum phenomena. Now, if we take the view that there is one standard of measurement for the proton and another for the electron, we can do without the introduction of a second mass, and retain one mass for both electron and proton. But we introduce two scales of measurement and so do not alter the number of constants. Some may prefer to introduce different masses directly rather than different scales, but there is more in the change than this, for the metrical view has certain other advantages. We can only decide which assumption to adopt when we discover that one idea is more fruitful than the other.

We know from the general theory of relativity that the space near a proton must be much more strongly curved than that near an electron and we can readily appreciate that the metric of space also might be markedly different in the two cases.

This point may be explained by reference again to the world lines of the proton and electron. These may be regarded as made of elements of lengths h/M_0c and h/m_0c. We have, as it were, two scales, one more finely divided than the other. Now the view under consideration is that this difference in magnitude is only apparent, the length h/M_0c in the proton space being regarded as equivalent physically to h/m_0c in the electron space. We have a kind of squeezing together of space in the proton relatively to that in the electron. The reason for this statement of equivalence of these two lengths is that in a parallel displacement in the proton region from one end of the element h/M_0c to the other we have the same change in length per unit length as in the electron space from one end of the element h/m_0c to the other.

If we judge the phenomena concerning electrons and protons from the same point of view and apply the same metrical considerations to each we shall conclude that a proton moves more slowly than an electron under the same conditions and so estimate its acceleration at too low a value and attribute to it greater inertia. We ought to regard the unit in the proton space not as of length h/M_0c but as h/m_0c, since the unit of length means physically so much more in that space. Thus our constant α is $\pm e/m_0c$, but with the change of sign we must change our metric.

There is another point. If this view is right the proton space is a kind of miniature of the electron space, and we should expect the radii of the two bodies to appear in the ratio of the units of the scales, i.e. inversely as their masses. This is in fact what is assumed about these bodies for, as we have seen,

$$r_0 = k\frac{e^2}{m_0 c^2} \text{ and } R_0 = k\frac{e^2}{M_0 c^2}.$$

This view is offered as no more than a suggestion of a new way of looking at one of the problems occupying us nowadays.

DISCUSSION

Prof. W. WILSON: Dr Flint's paper contains a further application of a principle which he proposed some time ago; he has already made some important applications of it, one of the most interesting being an estimation of the upper limit for the atomic numbers of the elements. I think it is no mere coincidence that this turns out to be about 97, while the atomic number of uranium is 92. It is hardly surprising that the distance h/m_0c should have a special significance. A similar expression, namely h/mv, where mv is the momentum of the electron, is well known to represent the experimentally determined value of the wave-length of the waves associated with the electron.

As Dr Flint remarks, his principle differs from the Bohr-Heisenberg uncertainty principle which asserts that the errors, $\triangle q$ and $\triangle p$, in the determination of the position and momentum of one electron, are governed by the relation

$$\triangle q \, \triangle p \risingdotseq h.$$

I feel some doubt as to whether the latter principle is correct. It is true, of course, that any device for accurately determining the position of an electron at some instant will cause it to suffer a violent change of momentum; but what reason prohibits us from employing a device which gives us accurately the velocity (only) of the electron and, *while this is in operation*, from employing some other device, which is adapted to determine its position accurately?

The asymmetry in the masses of the electron and proton is one of the most provoking of physical facts. It is one of the things we should like to be able to trace back to some still more fundamental fact, and Dr Flint's tentative suggestions are of great interest.

Mr J. E. CALTHROP questioned whether the author was justified in adding together the masses of the proton and electron in his calculation, since there would be a change of energy involved in their near approach.

Mr I. O. GRIFFITHS. I listened with interest to Dr Flint's account of his deduction of a fundamental length connected with the electron. In Fürth's paper, to which he refers, a value of a certain expression is obtained which is 2000 times the accepted value of the radius of the electron. As this expression involves the mass of the electron linearly, it was perhaps natural to introduce the atomic mass in addition to the electronic mass, thus introducing a factor of approximately 1840 which goes some way to counterbalance the factor of 2000. The agreement which is claimed would however have been more striking if it could have been obtained with a simpler fraction than 15/32 for k.

Dr W. N. BOND. I should like to express my appreciation of this further attempt by Dr Flint to relate theoretically experimental constants that were formerly unrelated. The comparative smallness of the number 1838, where numbers of the order 10^{10} and 10^{-10} may be concerned, indicates that the relationship might be expected to be calculable theoretically. The calculation given in the paper deduces the relationship in a very interesting way, which I am not in a position to criticise. As regards the numerical value, according to Birge* the ratio M_0/m_0 has two values. In any case, numerical agreement to nearer than one part in 1000 cannot be significant, as it is limited by the experimental errors in the value of hc/e^2. Apparently the experimental value $2\pi\,(137{\cdot}29 \pm 0{\cdot}11)$ has been used by the author, not the value $2\pi \times 137$ suggested by Eddington. Personally I feel that there is now enough evidence to accept the latter value. However, all lack of exact numerical agreement could be attributed to the uncertainty as to the value of k; and the doubt about this value seems the only inelegant part of the paper.

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* R. T. Birge, *Phys. Rev. Supp.* 1, 47, 48 (1929).

Capt. C. W. Hume. In attributing the difference in mass between the proton and electron to a change of metric, the author has invoked the increase in curvature of space-time in the neighbourhood of the proton. But is not the difference of mass already implicit in the difference of curvature? The proposed change seems to be only a verbal one.

Mr T. Smith said that he felt the greatest admiration for those developments of modern physics which skirted the border-line between the physical and the psychical, but he felt that a dictionary of the terms employed was badly needed. The author has used a phrase "physically equivalent," for instance, which the speaker was quite unable to understand.

Author's reply. It is not necessary to go into detail in this case with regard to the process of union of an electron and proton to form radiation. We must for our present purpose take account of the whole mass concerned, wherever it may be distributed.

The reference to the curvature in the neighbourhood of the proton was only by way of illustration; it is not desired to draw any conclusion from it, but only to point out that, as there are such great changes in the geometry in this region of space, we need not be surprised to find a metrical change also.

The meaning intended in the use of the term "physically equivalent" was that any change taking place in the electron region over the space of one unit length appropriate to this region may be regarded as equal to that which takes place in the protonic region along the unit appropriate to it. We may think of an observer attached to the electron and compare his experience with that of an observer attached to the proton. In the same electromagnetic field the experience of each would be the same along the units appropriate to them.

A HARMONIC ANALYSER

By J. HARVEY, A.R.C.S., B.Sc., City and Guilds Engineering College

Communicated by Dr E. Mallett January 28, 1930. *Read and discussed March* 14, 1930.

ABSTRACT. The paper describes a device for finding a given number of the Fourier components of a function by means of successive weighings.

§ 1. INTRODUCTION

FOURIER'S theorem states that if a function $f(x)$ is periodic in intervals of x equal to $2l$, then $f(x)$ can be expressed as the series

$$a_0 + a_1 \cos \frac{\pi}{l} x + a_2 \cos \frac{2\pi}{l} x + \dots$$

$$+ b_1 \sin \frac{\pi}{l} x + b_2 \sin \frac{2\pi}{l} x + \dots,$$

where $a_0, a_1, a_2, \dots b_1, b_2 \dots$ are constant coefficients.

If the form of the function $f(x)$ is not given, but the lengths $y_0, y_1, y_2, \dots y_{2n-1}$ of the $2n$ ordinates of the curve $y = f(x)$ for the values $0, \frac{l}{n}, \frac{2l}{n}, \dots (2n - 1)\frac{l}{n}$ of x are known, a finite series

$$a_0 + a_1 \cos \frac{\pi}{l} x + a_2 \cos \frac{2\pi}{l} x + \dots + a_n \cos n \frac{\pi}{l} x$$

$$+ b_1 \sin \frac{\pi}{l} x + b_2 \sin \frac{2\pi}{l} x + \dots + b_{n-1} \sin (n - 1) \frac{\pi}{l} x$$

of $2n$ terms can be found representing a function $\phi(x)$, which has the same values $y_0, y_1, y_2, \dots y_{2n-1}$ as $f(x)$ for the corresponding values of x.

Then
$$a_0 = \frac{1}{2n} [y_0 + y_1 + y_2 + \dots + y_{2n-1}],$$

$$a_r = \frac{1}{n} \left[y_0 + y_1 \cos \frac{r\pi}{n} + y_2 \cos \frac{2r\pi}{n} + \dots + y_{2n-1} \cos (2n - 1) \frac{r\pi}{n} \right],$$

where
$$r = 1, 2, \dots (n - 1).$$

$$a_n = \frac{1}{2n} [y_0 + y_1 \cos \pi + y_2 \cos 2\pi + \dots + y_{2n-1} \cos (2n - 1) \pi],$$

$$b_r = \frac{1}{n} \left[y_1 \sin \frac{r\pi}{n} + y_2 \sin \frac{2r\pi}{n} + \dots + y_{2n-1} \sin (2n - 1) \frac{r\pi}{n} \right],$$

where
$$r = 1, 2, \dots (n - 1)^*.$$

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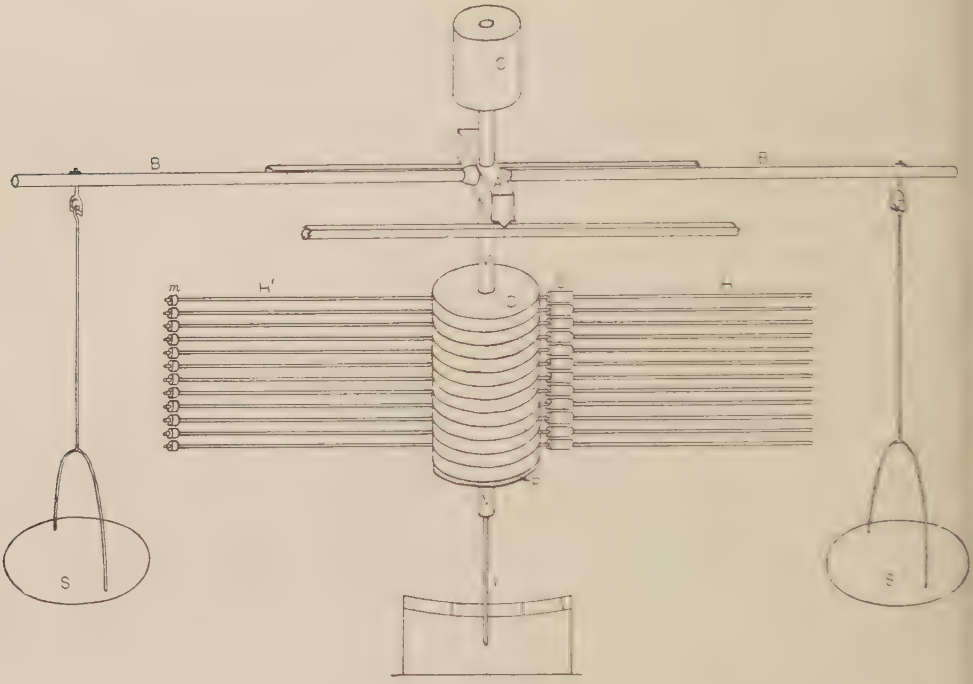
* See Whittaker and Robinson, *Calculus of Observations*.

By means of the harmonic analyser here described the summation of the series for the coefficients when $2n = 12$ can be effected. On the principle employed an instrument may be constructed for evaluating the series for the coefficients corresponding to any value of $2n$.



§2. THE INSTRUMENT

From a bar A, supported on horizontal knife-edges, a bar V is suspended which can rotate in a vertical plane called the principal plane, perpendicular to the knife-edges. Along the bar V there is a group of twelve ($= 2n$) circular discs D coaxial with the rod and supported by a plate P rigidly fixed to V. Each disc is capable of rotation about V. Fixed to each disc and projecting radially in opposite directions are two similar bars H and H'. The construction is such that any one disc may be turned about V into a position in which its bars H and H' make with the principal plane any angle which is a multiple of 30° ($= \pi/n$)*.

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* There are several methods of achieving this. One such is as follows: On the upper face of the plate P and of each disc, twelve radial grooves at angles of 30° with each other are cut. In a small cylindrical hole parallel with V cut in each disc upwards from the lower face, a plunger plays against a fine spring at the upper end. The lower end of the plunger is wedge-shaped and can enter any one of the twelve grooves in the disc beneath and give a slight check to the rotation of one disc with respect to the other at every 30°. The check is just sufficient to keep the discs, with their bars when loaded, relatively fixed during free oscillations of the whole system about the knife-edges. The plungers are placed in such positions in the discs that all the bars H and H' can be brought together into the principal plane. The rotation of a bar from this plane through an angle which is a multiple of 30° can then be made.

On the bar H attached to each disc is a "unit mass" U symmetrically disposed about the axis of the bar, and having sufficient friction with the bar to keep it in any position in which it has been placed. When a unit mass is situated in a certain position near the disc, as in the diagram, called the zero position, the bar V being vertical, the force of gravity on the unit mass produces a moment about the knife-edges. This moment is counter-balanced by that produced by a mass m fixed to the bar H' opposite. Hence when the unit masses on all the bars are at zero positions, the system with the bars in any relative horizontal positions will be in equilibrium about the knife-edges and V will be vertical. The vertical position of V is indicated by the coincidence of the prolongation of V, as an index I, with the zero of a fixed scale.

Fixed to the horizontal bar A carrying the knife-edges, at right angles to them and to the bar V, are two similar arms B of the same length, opposite each other, constituting a beam as in a balance for weighing. From these arms are hung scale-pans, S, which balance with respect to the knife-edges. A counterpoise C movable on the bar V can be adjusted in position so as to give sensitivity to the system.

On the base of the supporting frame there is a mechanism by which the bar V carrying the discs and bars is fixed or released: by a horizontal movement of a lever two jaws are opened or closed on the bar, while a vertical movement of the lever in a certain position locks or unlocks the jaws.

§ 3. DETERMINATION OF THE COEFFICIENTS

The twelve unit masses are moved along the bars, taken in order, from the zero positions through distances $y_0, y_1, y_2, \dots y_{11}$, corresponding to the lengths of twelve equidistant ordinates of the curve $y = f(x)$ in the range $x = 0$ to $x = 2l$. The bars are then brought into the principal plane so that lengths representing positive ordinates are in one sense from V while those representing negative ordinates are in the opposite sense. The bar V will then, in general, be deflected from the vertical position, positively or negatively. The vertical position is restored by placing mass in one of the scale-pans. The moment of the weight of this mass about the knife-edges is equal to the sum of the moments of the weights of the unit masses, which is

$$y_0 + y_1 + y_2 + \dots + y_{11}.$$

Hence the mass added to the scale pan is a measure of a_0.

In general, to obtain a coefficient a_r, the bars in the order referred to are turned so as to make angles 0°, $r \times 30°$, $r \times 60°, \dots r \times 330°$ respectively with the principal plane. Mass is then added to one of the scale-pans to maintain the bar V in the vertical position. The weights of the unit masses now exert about the knife-edges the moment

$$y_0 + y_1 \cos(r \times 30°) + y_2 \cos(2r \times 30°) + \dots + y_{11} \cos(11r \times 30°).$$

Hence the mass in the scale-pan is a measure of a_r.

To obtain the coefficient b_r the bars are kept in the same relative positions as for a_r, and the whole system is turned through a right angle about the bar V. If V is maintained vertical by the placing of mass in one of the scale-pans, the moment about the knife-edges of the weights of the unit masses is now

$$y_1 \sin (r \times 30^\circ) + y_2 \sin (2r \times 30^\circ) + \dots + y_{11} \sin (11r \times 30^\circ),$$

and the mass in the scale-pan is a measure of b_r.

On a model actually constructed each unit mass is of 50 grams and has a range of movement of 16 cm. on the bar H. Ordinates may be set off by callipers to within 0·5 mm. Each arm B is 30 cm. long, so that a movement of a unit mass through 0·5 mm. along the bar produces a moment about the knife-edges which can be equilibrated by a mass of 0·08 gram approximately in the scale-pan. The instrument can be made sensitive enough to weigh to 0·2 gram. If the ordinates be measured along the bars in centimetres, the mass placed in a scale-pan to maintain the bar V in the vertical position should be divided by 10 to give the coefficient sought, except in the cases of a_0 and a_6 when the mass should be divided by 20.

§ 4. ACCURACY

To test the accuracy of the model, the twelve ordinates of the curve

$$y = 3 - \tfrac{9}{2} \cos x - 3 \cos 2x + \tfrac{3}{2} \cos 3x$$
$$+ \tfrac{15}{4} \sin x + \tfrac{9}{4} \sin 2x + \tfrac{3}{4} \sin 4x$$

for the values 0, $\pi/6$, $2\pi/6$, ... $11\pi/6$ of x were calculated. These ordinates were set off along the bars H of the instrument and the coefficients of the cosines and sines in the given formula estimated. The ordinates were reset and the process repeated. The results are as shown in the table.

Coefficient	By calculation from series	By instrument (1)	By instrument (2)
a_0	3·00	2·94	2·93
a_1	−4·50	−4·44	−4·43
b_1	3·75	3·73	3·68
a_2	−3·00	−2·94	−2·97
b_2	2·25	2·20	2·23
a_3	1·50	1·47	1·48
b_3	0·00	0·03	0·01
a_4	0·00	0·00	0·00
b_4	0·75	0·75	0·77
a_5	0·00	0·01	0·01
b_5	0·00	0·00	0·01
a_6	0·00	0·00	0·01

The instrument should be sufficiently accurate for the analysis of many types of geophysical and meteorological phenomena.

5. EXTENSION TO MORE THAN TWELVE FOURIER COMPONENTS

An instrument consisting of twelve discs and bars may be employed to give the oefficients of the terms of a sine and cosine series for a number $2n$ which is a ultiple of twelve, if checks are provided so that the discs can be turned relatively hrough corresponding angles. Thus when $2n = 24$, twenty-four equidistant rdinates $y_0, y_1, y_2, \dots y_{23}$ of the given curve over the range of periodicity are easured, and relative movements of one disc over another through multiples of 5°, or $= \pi/12$, must be made. For the determination of a coefficient a_r, the unit asses on the twelve bars taken in order are moved distances $y_0, y_1, y_2, \dots y_{11}$ rom the zero positions. The bars are then turned through angles

$$0^\circ, (r \times 15^\circ), (2r \times 15^\circ), \dots (11r \times 15^\circ)$$

espectively from the principal plane. A mass M_1 is added to a scale-pan to main-ain the bar V in the vertical position. Next, the unit masses are moved so that hey are at distances $y_{12}, y_{13}, \dots y_{23}$ along the bars taken in the same order as before rom the zero positions, and the bars are turned so that they make angles

$$(12r \times 15^\circ), (13r \times 15^\circ), \dots (23r \times 15^\circ)$$

espectively with the principal plane. Mass M_2 is added to a scale-pan to main-ain V vertical. Then $M_1 + M_2$ is a measure of a_r. The coefficient b_r may be btained similarly from the masses required to maintain V vertical when each f the two systems of bars, as arranged for a_r, is turned through a right angle.

§ 6. ACKNOWLEDGMENTS

The author acknowledges his indebtedness to Mr E. A. Nehan, Mechanics and Mathematics Laboratory, City and Guilds (Engineering) College, South Kensington, for constructing the model illustrated, and for the valuable advice and help which he has at all times generously given.

DISCUSSION

Mr J. H. Awbery congratulated the author on his ingenious invention and asked how much time it would require for an analysis, as compared with the time taken by arithmetical methods. The data given in the paper showed that the results were sufficiently accurate for practical purposes.

Mr J. E. Calthrop. Speaking from the point of view of a teacher of physics, I should like to congratulate the author on a piece of apparatus which should be of great use and interest in a laboratory for teaching purposes. I suppose the use of callipers could be eliminated by suitable graduation of the arms carrying the movable weights.

The President said that the instrument, by its direct attack on Fourier analysis, would give students a very clear view of the principles of the subject. Many persons must have interested themselves in the problem, but probably the instrument shown at the meeting was the first that had been actually constructed, at all events with such excellent workmanship and accuracy.

AUTHOR'S REPLY. The bars carrying the movable weights could be graduated, but it would, I think, be more difficult to set off the ordinates along the bars by the graduations than by the aid of callipers. The pair of callipers, with its vernier, can be turned in the hand towards the light and the length of an ordinate set off on it accurately and rapidly.

In the estimation of the coefficients of the function given in the table in the paper, the time taken by using the instrument was under twenty-five minutes, a single weighing being made for each coefficient, and I do not think that much time would be saved by this method in comparison with certain arithmetical methods in the case of twelve ordinates.

A METHOD OF COMPARING SMALL MAGNETIC SUSCEPTIBILITIES

By R. A. FEREDAY, B.Sc., East London College.

Communicated February 26, 1930. *Read and discussed March* 14, 1930.

ABSTRACT. A method of the non-uniform field type is described. It possesses the specific advantage that the force exerted on the specimen is constant over a wide region of the field, so that a careful setting to a position of maximum force is rendered unnecessary.

§ 1. INTRODUCTORY

THE present communication gives an account of a method which has been developed at the suggestion of Prof. C. H. Lees, and in consultation with him, for the determination of small magnetic susceptibilities, and departs considerably from general practice in measurements of this type. The apparatus was needed for an investigation in which a high sensitivity was required when a specimen of but a few milligrams was being used, and for this reason, and since relative measurements only were required, a method of the non-uniform field type was chosen.

§ 2. THE NON-UNIFORM FIELD METHOD

The force due to a field H, acting in any direction s, on a small body of volume v and susceptibility κ, that of the surrounding medium being κ_0, is given by

$$f_s = (\kappa - \kappa_0)\, vH \frac{\partial H}{\partial s} \qquad \text{......(1).}$$

Thus, if two bodies are brought in succession to the same part of the field, and the forces on them are compared, we shall have:

$$\frac{\kappa_1 - \kappa_0}{\kappa_2 - \kappa_0} = \frac{f_1}{f_2} \qquad \text{......(2).}$$

In the derivation of (1), it is assumed that the force on all parts of the body is the same, a condition which it is almost impossible to realise in practice with a specimen of reasonable size. Equation (2) will, however, hold accurately for larger specimens, provided they have the same size and shape, and are brought to the same position in the field*. The method is therefore much better suited for relative than for absolute measurements, but each specimen must be placed at the same point in the field, and this constitutes the chief difficulty of the method.

Curie† and later observers used the magnetic field between two inclined poles; the direction of the field gradient, and hence that of the force exerted on the specimen, being at right angles to that of the field itself. A curve is drawn showing the variation of force along the direction of the gradient, and it is found that at a certain point it passes through a fairly narrow maximum, on each side of which it falls off sharply. In the comparing of susceptibilities, each specimen is placed at this point of maximum force, and to secure this, the electromagnet producing the field is usually mounted on some form of carriage, so that it can be displaced relatively to the specimen until the latter is in the desired position. It is, however, by no means easy to impart micrometric movements to a heavy electromagnet, especially when the specimen is suspended in the interpole gap and must not be unduly disturbed.

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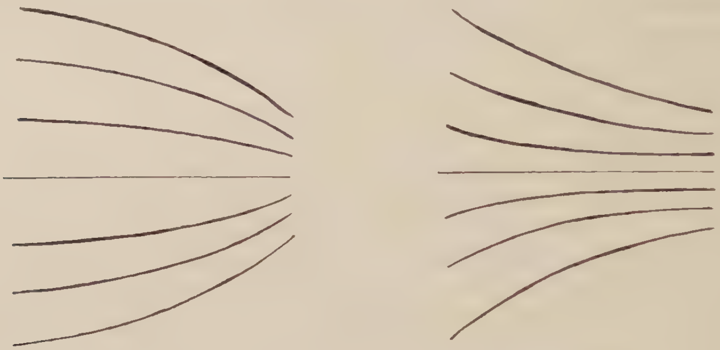
* Cf. Stoner, *Magnetism and Atomic Structure*, p. 39.

† *Ann. Ch. Ph.* 7, 289 (1895).



(*a*) (*b*)

Fig. 1. Lines of force in converging fields.

Another form of the non-uniform field method was introduced by Wiedemann* but has not found favour with modern workers. In this method, the field, its gradient, and the force to be measured, are all directed along the axis of symmetry of the field, which is produced by the pointed pole of an electromagnet. A suitable field must be of one of the types of Fig. 1, which represents sections of the field by a plane passing through the axis of symmetry.

In the case of a field of type 1 (*a*), both the field intensity and its gradient increase from left to right of the figure; hence the force exerted on a body placed on the axis will be very much greater at the right than at the left of the figure, and if measurements were made in such a field it would be necessary to attend very carefully to the setting of the specimens. With a field of type 1 (*b*), on the other hand, the field decreases as its gradient increases, so that the variation of their product along the axis will be small. If these opposite variations could be made to balance exactly, so that $H \times \partial H/\partial x$ had a constant value at all points of the axis, then the force exerted on a given specimen would be independent of its position so long as it remained on the axis, and the great experimental difficulty of setting the specimen would entirely disappear. It is on this conception that the method to be described is based.

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* *Pogg. Ann.* **126**, 1 (1865).

§ 3. THE MAGNETIC FIELD

In an attempt to realise a field satisfying the condition $H.\partial H/\partial x =$ constant along the axis of symmetry OX, the assumption was made that the pole pieces of the magnet could be used with such a field that the permeability of their material would be high; the pole faces would then very nearly be magnetic equipotential surfaces. The problem thus reduces to the determination of equipotential surfaces in a field satisfying the above condition. Further, since the field is symmetrical about an axis, it is only necessary to determine the curves of section of the equipotentials by a plane through this axis, and in the sequel, the term "equipotential" is used to describe such curves. Two systems of co-ordinates are used in the plane, one a polar system, in which the axis of symmetry is $\theta = 0$, and the other a Cartesian system, the axis of symmetry being taken as OX and a perpendicular at the origin as OY. In each case, the centre of the interpole gap is taken as origin. The equipotentials were first determined by a simple semi-graphical method, and the results were used to design a pair of pole pieces. With these pole pieces in the apparatus to be described, numerical data were obtained which served for guidance in the more accurate analysis, which is given in the next paragraph*.

The important constants determined experimentally were:

Field at origin (H_0)	1000 gauss.
$H.\partial H/\partial x$	3×10^5.

Since the field is symmetrical about the axis, the magnetic potential at the point (r, θ) may be expressed as

$$V = V_0 + A_1 r P_1(\cos\theta) + A_2 r^2 P_2(\cos\theta) + A_3 r^3 P_3(\cos\theta) + \dots \quad \dots\dots(3).$$

For any point on the axis of symmetry, the Legendre polynomial $P_n(\cos\theta)$ is equal to unity, so that the potential at any point on the axis is given by

$$V = V_0 + A_1 r + A_2 r^2 + A_3 r^3 + \dots \quad \dots\dots(4).$$

The field along the axis at the point $(r, 0)$ is therefore given by

$$\begin{aligned} -H = \partial V/\partial r &= A_1 + 2A_2 r + 3A_3 r^2 + 4A_4 r^3 + \dots \\ &= H_0 + 2A_2 r + 3A_3 r^2 + 4A_4 r^3 + \dots \quad \dots\dots(5) \end{aligned}$$

(H_0 being the field at the origin),

therefore

$$H^2 = H_0^2 + 4H_0 A_2 r + (6H_0 A_3 + 4A_2^2) r^2 + (8H_0 A_4 + 12A_2 A_3) r^3 + \dots$$

$$\frac{\partial}{\partial r}(H^2) = 4H_0 A_2 + 2(6H_0 A_3 + 4A_2^2) r + 3(8H_0 A_4 + 12A_2 A_3) r^2 + \quad \dots\dots(6).$$

Now we require to satisfy the condition that $H.\partial H/\partial r$, i.e. $\frac{1}{2}.\partial H^2/\partial r$ shall be independent of r, having a constant value of, say, c. This condition will be satisfied if the coefficients of powers of r in (6) all vanish; i.e. if

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* I am indebted to Prof. Lees for suggesting this method of attack, and for assistance in its development.

$$\frac{\partial}{\partial r}(H^2) = 4H_0A_2 = 2c;\ \text{i.e. } A_2 = \frac{c}{2H_0},$$

$$6H_0A_3 + 4A_2^2 = 0;\ \text{i.e. } A_3 = -\frac{2A_2^2}{3H_0} = -\frac{c^2}{6H_0^3},$$

$$8H_0A_4 + 12A_2A_3 = 0;\ \text{i.e. } A_4 = -\frac{3A_2}{2H_0}\cdot A_3 = +\frac{c^3}{8H_0^5}.$$

Substituting these values of the coefficients A_1, A_2, etc., in (3), we have, as the expression for the potential at the point (r, θ) in a field which satisfies the required condition along the axis,

$$V = V_0 + H_0rP_1(\cos\theta) + \frac{c}{2H_0}r^2P_2(\cos\theta) - \frac{c^2}{6H_0^3}r^3P_3(\cos\theta) + \ldots \qquad \ldots\ldots(7),$$

$$\frac{V - V_0}{H_0} = rP_1(\cos\theta) + \frac{c}{2H_0^2}r^2P_2(\cos\theta) - \frac{c^2}{6H_0^4}r^3P_3(\cos\theta) + \ldots \qquad \ldots\ldots(7\text{ a}).$$

Substituting the numerical values characterising the experimental apparatus, i.e. $H_0 = 1000$, $c = 3 \times 10^5$, we have

$$\frac{c}{2H_0^2} = 0{\cdot}15;\quad \frac{c^2}{6H_0^4} = 0{\cdot}015.$$

The series will converge least rapidly when r and $\cos\theta$ have their maximum values. The pole gap being, in the apparatus, 3·4 cm., r cannot exceed 1·7, so that when the series of (7 a) is converging least rapidly, its terms will have values

$$1{\cdot}7 + 0{\cdot}43 - 0{\cdot}073 + \ldots.$$

It is thus clear that for an apparatus not too different from the one tested, the value of $(V - V_0)/H_0$ may be expressed by the first two terms of (7 a), without serious error.

We thus have

$$\frac{V - V_0}{H_0} = rP_1(\cos\theta) + \frac{c}{2H_0^2}r^2P_2(\cos\theta) = r\cos\theta + \frac{c}{4H_0^2}r^2(3\cos^2\theta - 1) \quad \ldots\ldots(8).$$

If different values are taken for V in this equation, we have a series of equations to the corresponding equipotentials. It will be convenient to use the same equation, referred to the Cartesian system of co-ordinates; its form in this case is

$$\frac{V - V_0}{H_0} = x + \frac{c}{4H_0^2}(2x^2 - y^2) \qquad \ldots\ldots(8\text{ a}).$$

The equation may also be expressed in a third form. Suppose that the equipotential for some particular value of V cuts the axis of symmetry at (X, O). Then we must have

$$\frac{V - V_0}{H_0} = X + \frac{c}{4H_0^2}(2X^2 - 0),$$

so that the general equation may be written

$$X + \frac{c}{2H_0^2} X^2 = x + \frac{c}{2H_0^2} x^2 - \frac{c}{4H_0^2} y^2.$$

If we write $\frac{V - V_0}{H_0} = a$, $\frac{c}{4H_0^2} = b$, then the different forms of the equation to the equipotentials become respectively

$$a = r . \cos\theta + br^2 (3 \cos^2\theta - 1) \qquad \text{......(8)},$$

$$a = x + b(2x^2 - y^2) \qquad \text{......(8 a)},$$

$$(X + 2bX^2) = (x + 2bx^2) - by^2 \qquad \text{......(8 b)},$$

from which successive equipotentials are determined by giving successive values to either a or X.

The equations just derived define equipotential surfaces of infinite extent. In fashioning a pole face to be an equipotential surface it is, however, necessary to use a limited area. The condition to be satisfied is that if the field is produced between two poles shaped as equipotential surfaces according to the above equations, and separated by the appropriate distance, then their areas must bear such a ratio to each other that the same total flux passes through each.

We shall satisfy this condition by terminating the sections of the equipotential surfaces at points which are on the same line of force. We have the equation to the equipotential (equation 8 b),

$$a = x + b(2x^2 - y^2),$$

$$\therefore\ 0 = dx + b(4x . dx - 2y . dy),$$

$$(4bx + 1)\, dx = 2by . dy.$$

A system of normal curves (lines of force) must satisfy

$$(4bx + 1)\, dy = -2by . dx,$$

$$\frac{dy}{2by} + \frac{dx}{4bx + 1} = 0,$$

$$(2b)^{-1} \log . y + (4b)^{-1} \log (4bx + 1) = \text{const.}$$

$$2 \log . y + \log (4bx + 1) = \text{const.}$$

$$y^2 (4bx + 1) = \text{const.}$$

Thus, if one equipotential is terminated at the point (x_1, y_1), the other must be terminated at a point (x_2, y_2), related by the expression

$$y_1^2 (4bx_1 + 1) = y_2^2 (4bx_2 + 1),$$

$$\frac{y_1^2}{y_2^2} = \frac{4bx_1 + 1}{4bx_2 + 1} \qquad \text{......(9)}.$$

If the desired interpole gap $(2X)$, and the radius of the larger pole (y_1) are known, then equations (8 b) and (9) suffice to determine a pair of pole pieces for any assumed value of b.

§ 4. EFFECT OF FINITE SIZE OF SPECIMEN

Having determined a field distribution such that a point specimen placed on the axis experiences a force which is independent of its position on the axis, we have to find to what extent this constancy is affected when a body of finite size is considered.

For a finite body symmetrical about the axis of symmetry of the field, the resultant force will still be directed along this axis, but that portion of it which is due to the effect of the field on the outer parts of the specimen may vary with position along the axis, and thus destroy the constancy of the total force.

We have, from (8 a), the potential at any point (x, y);

$$V = V_0 + xH_0 + \frac{c}{4H_0}(2x^2 - y^2),$$

therefore the field, H_x, along a direction parallel to the axis OX is

$$H_x = H_0 + \frac{c}{H_0} . x,$$

$$\frac{\partial H_x}{\partial x} = \frac{c}{H_0}.$$

Thus the product $H . \partial H/\partial x$, which determines the force on the specimen, should be quite independent of y. This approximation is not close enough for the present purpose, and a more accurate calculation must be made in which a further term of (7) is taken into account. Thus we have

$$\begin{aligned} V &= V_0 + H_0 r P_1(\cos\theta) + \frac{c}{2H_0} r^2 P_2(\cos\theta) - \frac{c^2}{6H_0^3} r^3 P_3(\cos\theta) \\ &= V_0 + H_0 r . \cos\theta + \frac{c}{4H_0} r^2 (3\cos^2\theta - 1) - \frac{c^2}{12H_0^3} r^3 (5\cos^3\theta - 3\cos\theta) \\ &= V_0 + H_0 x + \frac{c}{4H_0}(2x^2 - y^2) - \frac{c^2}{12H_0^3}(2x^3 - 3xy^2), \end{aligned}$$

$$H_x = H_0 + \frac{c}{H_0} x - \frac{c^2}{2H_0^3} x^2 + \frac{c^2}{4H_0^3} y^2,$$

$$\begin{aligned} H_x^2 = H_0^2 + \frac{c^2}{H_0^2} x^2 + \frac{c^4}{4H_0^3} x^4 + \frac{c^4}{16H_0^6} y^4 + 2cx - \frac{c^2}{H_0^2} x^2 + \frac{c^2}{2H_0^2} y^2 - \frac{c^3}{H_0^4} x^3 \\ + \frac{c^3}{2H_0^4} xy^2 - \frac{c^4}{4H_0^6} x^2 y^2, \end{aligned}$$

$$\frac{\partial}{\partial x}(H_x^2) = \tfrac{1}{2} H \frac{\partial H}{\partial x} = 2\frac{c^2}{H_0^2} x + \frac{c^4}{H_0^3} x^3 + 2c - \frac{2c^2}{H_0^2} x - \frac{3c^2}{H_0^4} x^2 + \frac{c^3}{2H_0^4} y^2 - \frac{c^4}{2H_0^6} xy^2,$$

$$\frac{\partial}{\partial y}\left\{\tfrac{1}{2} H \frac{\partial H}{\partial x}\right\} = \frac{c^3}{H_0^4} y - \frac{c^4}{H_0^6} xy,$$

$$\frac{\partial}{\partial y}\left\{H \frac{\partial H}{\partial x}\right\} = \frac{2c^3}{H_0^4} y \left(1 - \frac{c}{H_0^2} x\right).$$

Substituting numerical values for the present experimental apparatus, we have at the centre of the gap ($x = 0$)

$$\frac{\partial}{\partial y}\left\{H\frac{\partial H}{\partial x}\right\} = \frac{\partial c}{\partial y} = 54000y.$$

As an example, consider a point at 5 mm. from the axis. The mean gradient over the interval is that at $y = 0{\cdot}25$, which is 13500. The change in force is $y.\partial y/\partial x = 6250$. Since the value of c on the axis is $3{\cdot}10^5$, the percentage change in its value at 5 mm. is thus only of the order 5 per cent. Thus, the force exerted on the outer part of a specimen of 1 cm. diameter is 5 per cent. different from that exerted on the centre, so that the force exerted on the whole body will differ by only one or two per cent. from that on a point at its centre, and will thus be constant to within this degree of accuracy as the body is shifted to different parts of the axis.

It must be remembered that in practice the force at a distance from the axis will be seriously affected by "fringing" effects, unless the magnet poles are of large diameter. It is, however, clear from the foregoing analysis that if the specimen is small compared with this diameter it is quite unnecessary to take elaborate precautions to bring it to a definite position in the field, since $H.\partial H/\partial x$ will have a value which is appreciably constant over a very large region of the interpole gap.

§ 5. EXPERIMENTAL REALISATION

As has already been stated, a pair of poles has been designed by a less accurate method; these have been used with a simple torsion balance for susceptibility measurements, as it was found that their shape does not differ seriously from that indicated by the more accurate theory. The force obtained in the pole gap is not accurately constant, but the constant nature of the susceptibility measurements obtained shows that it does not vary seriously. As a precautionary measure, however, the specimens are set roughly to the same point on the axis when suscepti-

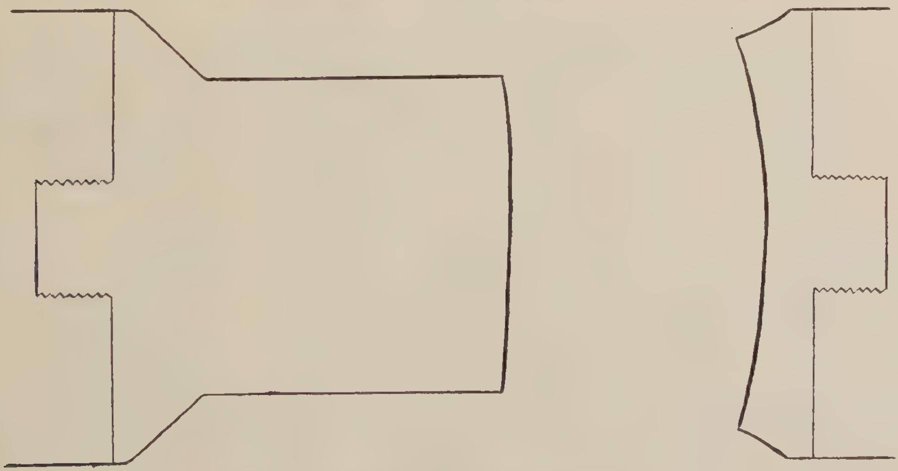

Fig. 2. Section of pole pieces (full size).

bilities are being compared; for this purpose, a small pointed plug has been let into the centre of one pole face, and the specimens are set to a standard distance from this.

Any attempt to realise the theoretical conditions more accurately has been postponed until experience of the method has been gained from a series of measurements with the present apparatus.

The pole pieces in use are shown in Fig. 2, from which it will be seen that the smaller one has been elongated in order to reduce the disturbing effect of the outer part of the pole core; this is necessary since the poles of the electromagnet are of equal diameters.

§ 6. APPARATUS

The electromagnet. Details of this are shown in Fig. 3. The coils have about 2500 turns of heavy-gauge copper wire, and can be supplied with current up to 10 amp. from accumulators. As will be seen from the figure, copper tubes, through which cold water can be circulated, are disposed so as to prevent heating of the poles or the air gap by the windings. A massive brass bar is bolted across the top of the magnet to keep the poles at a fixed distance apart, and this serves as a support for the torsion balance used in measuring the force on the specimen.

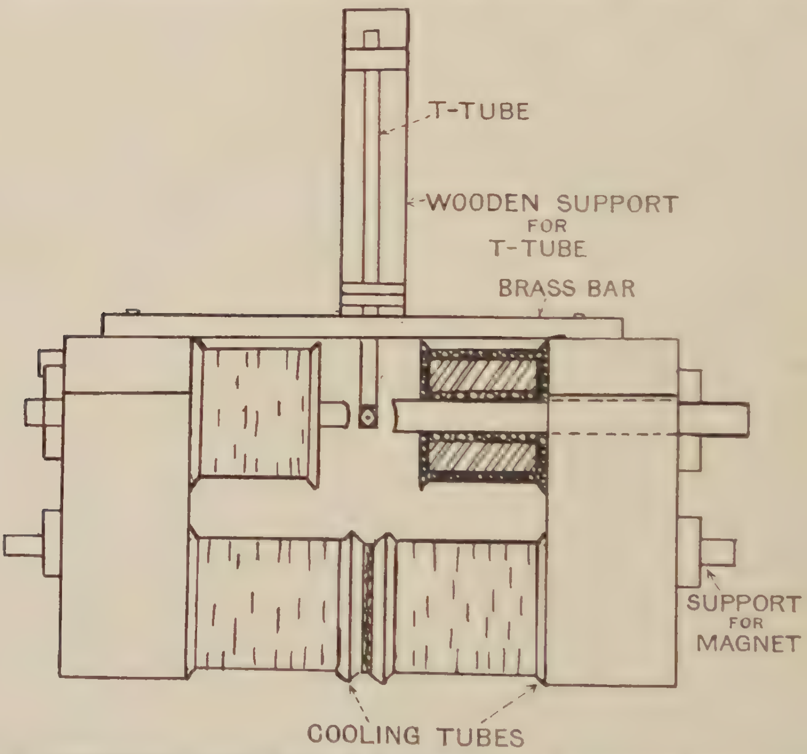

Fig. 3. Electromagnet (partly sectioned) showing method of supporting torsion balance.

The torsion balance. The whole of this system is contained in a T-tube of 1-in. glass tubing, each arm of which is about 12 in. long (Fig. 4); it is held in wooden clamps attached to the brass bar just mentioned. The tube is provided with a brass torsion head *H* fitting into its upper end, through which passes a brass rod *R* which is secured by the screw *G* and can be adjusted vertically to suit suspensions of different lengths. The lower end of this rod is filed down to a diametral plane to which the suspension is attached. To the lower end of the suspension, at *O*, is fixed the torsion arm *A*, which is made of glass rod about 0·5 mm. in thickness, except for a short thicker portion at *O*. The lengths of the two parts *AO*, *OA*, of the arm are about 5 cm. and 20 cm. respectively. At the end of the shorter part is a small glass cup *C*, which takes the specimen; at the other end there is a pointer *P* of very fine glass, which is used in conjunction with a reading microscope to measure deflections of the system in a manner previously described*.

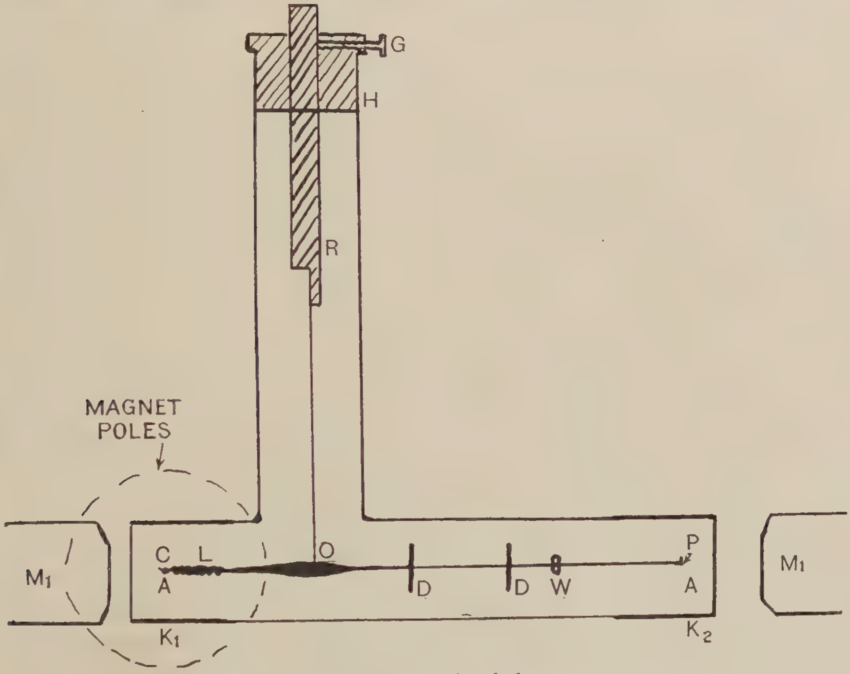

Fig. 4. The torsion balance.

The difference in weights of the two arms is balanced by a spiral of lead wire *L*, this material being chosen for its low susceptibility and high density. Small changes in balance of the system due to difference in weights of the specimens are compensated by shifting of the copper wire rider *W*.

The ends of the horizontal part of the tube are closed by two closely fitting paper caps K_1, K_2, provided with microscope cover slips as windows, through which microscope observations can be made without danger of error.

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* *Journ. Sci. Inst.* **6**, 302 (1929).

The substance examined is enclosed in a small glass bulb from 2 to 3 mm. in diameter which fits into the cup *C*. The clamps hold the T-tube at such a distance from the electromagnet that, when the tube is adjusted at the correct height, the cup lies on the axis of symmetry of the field.

Considerable difficulty was experienced in finding a suitable non-magnetic adhesive for the attachment of the suspension and lead counterweight, as any force exerted by the field on the adhesive, especially that used in attaching the suspension to the torsion arm, caused a deflection due to a motion of translation and not to a rotation about the suspension. Shellac varnish, applied thinly and "baked" by means of an electrically heated wire, was ultimately found satisfactory.

The apparatus proved so sensitive that for the observations made so far, on somewhat strongly magnetic salts, it has been found necessary to provide a bifilar suspension of relatively high restoring couple. A silk fibre is joined at each end to the torsion arm, and passes over an accurately turned and freely pivoted pulley at the lower end of the rod *R*. The difference in weights of the loaded and unloaded arm is so slight that the restoring couple can be considered as unaffected by the differing weights of the specimens. The provision of two light paper dampers *D*, *D*, sufficed to damp out pendular oscillations, which at first were very troublesome.

§7. EXPERIMENTAL PROCEDURE

Small glass bulbs with short narrow necks are used to hold the powdered specimens. An empty bulb is weighed and placed in the cup at the end of the torsion arm; the rider is adjusted so that the arm balances horizontally, and the two paper caps are placed on the ends of the T-tube. The tube is then slipped up or down in its wooden clamp until the bulb, observed through the microscope M_1 (Fig. 4), is seen to occupy a position on the axis of symmetry of the field; this adjustment is made with ample accuracy by eye. The torsion head is now turned until, with the magnet energised, the bulb occupies a position at a standard distance from the brass plug in one pole piece. These simple adjustments ensure that the bulb occupies, with sufficient accuracy, a standard position on the axis.

The pointer is now observed through the high power microscope M_2, and from observations of successive turning points of small oscillations, the positions of rest for no field and with the field applied are deduced. The deflections of the pointer never exceed 8 mm., corresponding to a movement of the bulb of 2 mm. These observations give the effect of the field on the empty bulb; they are repeated after the bulb has been filled with a powdered salt, and the effect on the salt alone is deduced by difference. The magnetising current is measured with a high grade moving-coil ammeter, and after adjustment to a standard value of 2·1 amp., the circuit is made and broken several times in order to obtain the true field corresponding to this current.

Generally, the weight of the salt is of the same order as that of the bulb containing it (about 10 mg.) and the deflection in the magnetic apparatus is about 40 times that due to the empty bulb. The weighings are conducted on a Bunge short-beam balance specially adjusted for high sensitivity in the weighing of masses of the order in question.

With such small quantities of material, it is essential to carry out the work with the utmost care, since a very slight trace of any impurity having a susceptibility differing appreciably from that of the material under examination would suffice to invalidate the reading. This was at first a source of much trouble, but as the technique of the work was mastered the difficulties disappeared. In order, however, to detect any trouble of this nature, susceptibility measurements were made on three or four specimens of each substance, and the mean was only accepted if the individual determinations were in satisfactory agreement. The values of the susceptibility deduced from observations on different specimens of the same substance rarely differed by more than 3 per cent.

§ 8. RESULTS AND CONCLUSIONS

In all the observations made up to the present time, pure heptahydrated nickel sulphate has been used as a standard of reference for relative measurements. Observations have been grouped, a separate calibration of the apparatus in terms of this salt being made for each group of four or five substances. As illustrating the constancy of the calibration, the results for four successive groups, extending over a period of eleven months, are given. The numbers tabulated represent the susceptibility of the sulphate in arbitrary units characteristic of the apparatus.

October 1927	750
November 1927	757
March 1928	772
June 1928	761

The extreme values differ by just under 3 per cent., and the constancy over the period of any one group was certainly of a much higher order, since precautions were taken to prevent disturbance of the apparatus over the period of each group, whereas adjustments and minor alterations were sometimes made between calibrations.

As work on the measurements is still in progress, the publication of the full results, comprising measurements on a wide range of nickel salts, is left to a later date; but since in two cases results of another worker are available, they are given here for comparison. Jackson* gives the following susceptibilities:

$$\mathrm{NiSO_4.7H_2O} = 16{\cdot}0 \times 10^{-6} \text{ at } 292{\cdot}2^\circ \text{ abs.}$$
$$\mathrm{NiSO_4(NH_4)_2SO_4.6H_2O} = 10{\cdot}7 \times 10^{-6} \text{ at } 286{\cdot}5^\circ \text{ abs.}$$

The present measurements give for the susceptibilities of these salts at room temperatures, in arbitrary units, 750 and 517 respectively. Taking Jackson's value for the sulphate as standard, this gives 11·0 × 10^{-6} as the susceptibility of the double salt, which is in satisfactory agreement with Jackson's own value.

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* *Phil. Trans.* A, **224**, 1 (1923).

The consistency of the results, and the ease with which they can be obtained, have proved to be entirely satisfactory, justifying the further development of the method. Work has accordingly been commenced on the design of an improved apparatus, in which it is hoped to realise the theoretically desirable field distribution more accurately. Attention is also being paid to other experimental details, with the object of developing an apparatus capable of giving precision results with a greater ease than has been possible hitherto.

§ 9. ACKNOWLEDGMENT

The work has been carried out in the Physics Department of East London College, under Prof. C. H. Lees, to whom I am indebted for much valuable advice, and general supervision of the work. I desire also to thank H.M. Department of Scientific and Industrial Research for financial assistance.

DISCUSSION

Dr D. Owen said that the author's method represented a definite advance on previous methods. It was an important achievement to have moved in the direction of making $H.\partial H/\partial x$ constant, for variations in this quantity admittedly gave rise to considerable errors in Curie's work. The speaker expressed a hope that the author's method would be used in a thorough repetition of investigations such as the effect of temperature on magnetisation. For this purpose it would be necessary to measure considerably lower values of the susceptibility than those referred to in the paper, and this appeared to demand that the value of the product $H.\partial H/\partial x$ should be made substantially higher than at present. Would it be feasible to design a magnetic circuit with a rapidly diverging field for this purpose?

Dr J. H. Shaxby: I also should like to congratulate the author on a very beautiful method of using a non-uniform field to give a uniform product. Is it possible to adapt the method to measure the susceptibility of fluids?

Mr J. Guild raised the question whether the treatment of end effects in § 4 of the paper covered all the demagnetising effects of the ends.

Author's reply. In reply to Dr Owen, I would point out that although no measurements have yet been made, with the apparatus described, of susceptibilities smaller than 10^{-15}, this is not due to any lack of sensitivity on account of field design; actually the value of $H.\partial H/\partial x$ is comparatively large, and by a simple modification of the torsion balance it will undoubtedly be possible to attain a very high sensitivity.

In attempting to increase the value of the product $H.\partial H/\partial x$ by making the field diverge more rapidly, one is faced with the difficulty that one of the poles must be made of small diameter; "fringing" effects then become more serious, and the theoretical conditions cannot be realised so accurately. Also, an increase in $\partial H/\partial x$ by this means, for a constant magnetising current and available gap, reduces H to some extent.

I have not used the apparatus to measure the susceptibilities of fluids, but there is no reason why it should not be used for that purpose.

The treatment of § 4 is intended as an investigation of the field configuration in the absence of the specimen. The correction to be applied for demagnetising effects of the ends of the latter would be of the order of only 0·01 per cent. and is therefore neglected.

THE EQUATIONS OF MOTION OF A VISCOUS FLUID IN TENSOR NOTATION

By C. N. H. LOCK, M.A., National Physical Laboratory.

Communicated by T. Smith, M.A., April 25, 1929, *and in revised form January* 18, 1930.
Read and discussed February 28, 1930.

ABSTRACT. The tensor notation, which is probably best known in its application in Einstein's four-dimensional theory of gravitation, can be used in its three-dimensional form in obtaining the general equations in many branches of mathematical physics. In the present paper the various steps in the proof of the equations of motion of a viscous fluid are translated into tensor form. An attempt has been made to develop the various formulae in outline from first principles, with references where necessary to a standard work, and also to give whenever possible a geometrical interpretation of the various vector quantities met with in the course of the argument. In particular the fundamental assumptions of Stokes, on which the viscous equations are based, are stated in the form of the relation between the stress vector across an element of area and the vector element of area.

The equations of motion in tensor notation are afterwards translated into ordinary notation for the special case of orthogonal curvilinear coordinates. These can be at once reduced to the commonly useful forms of spherical polars, ellipsoidal coordinates, etc. In a final section it is shown how the tensor equations can be used to deduce the equations of viscous flow in terms of Stokes stream function for motion symmetrical about an axis. The stream function is shown to be identical with the surviving component of the covariant vector potential of which the other two components vanish.

§ 1. INTRODUCTION

In all physical problems of a general nature and in hydrodynamics in particular, a considerable simplification arises if the various vector quantities are represented by a single symbol instead of the three components of the vector in Cartesian coordinates. The result of doing so is to ensure that none but vector expressions occur in the equations; on the other hand, the vector expressions (such as the vector product, divergence, curl, etc.) have to be manipulated by apparently arbitrary rules, many of which must be proved by returning to the Cartesian notation. The tensor notation, on the other hand, has the advantage that it makes use of none but the ordinary rules of algebra and the calculus, and, while the actual components of vectors are dealt with, it is possible by use of suitable notation and by a special convention (the summation convention*) to obtain the same economy in the use of symbols as with the ordinary vector notation.

In the present paper an attempt is made to develop the theory of the motion of a fluid in outline as far as the general equations of viscous flow by the use of the tensor notation in three dimensions. The actual proof of the equations of viscous flow is made to depend on the assumption that the resultant stress across an arbitrary element of area is a function of the first derivative tensor of the velocity vector, is linear in this derivative and vanishes with the element of area. The resulting function must be a vector function, and any properties of the fluid con-

* Appendix A.

tained in it must have the form of scalars, since the fluid can have no directional properties. The customary argument by transformation to the principal axes of the rate-of-strain ellipsoid is replaced by the assumption that the form adopted for the vector function representing the stress is the most general possible subject to the above conditions.

It would have been possible to give the whole of this argument in a form restricted to Cartesian coordinates independent of the particular system, and in fact the form of most of the actual expressions representing physical quantities would have been absolutely unaltered. It seemed however worth while to use the more general form applicable to general curvilinear coordinates, as by this means it was possible to use the results to deduce the equations in general orthogonal coordinates in ordinary notation.

An attempt has been made to develop the tensor calculus from first principles so far as it is required, though such proofs of the various steps as may be found by reference to standard works have for the most part been omitted. Reference is frequently made to *The Absolute Differential Calculus* by Levi-Civita (English translation)*. Definitions of the various physical quantities occurring in the equations of motion are given in some detail; after the proof of the equations of motion, the various tensor expressions, including the expressions for the stresses, are converted to general orthogonal coordinates in ordinary notation.

The method of eliminating the pressures, given in Lamb's *Hydrodynamics*†, is developed in tensor notation and expressed in terms of the vector potential. The result is used to develop the equation for the special case of motion symmetrical about an axis in orthogonal coordinates, in terms of Stokes' stream function ψ, by showing that ψ is identical with the third component of the covariant vector potential, the other two components being zero.

§2. THEORY OF TENSORS

The essential feature of the tensor notation is that it applies equally to any possible system of coordinates. Starting with any Cartesian system $X_i, (X_1, X_2, X_3)$, we take as the typical system of coordinates any three arbitrary functions of the X_i,

$$x_j = f_j(X_1, X_2, X_3).$$

These generalised coordinates will be in general both curvilinear and oblique; but the tensor formulae might be restricted if required to the special case of the transformation of Cartesian axes by displacement or rotation, which would lead to important simplifications in detail but would leave the general results unaltered.

In the tensor calculus we are concerned with transformations from any coordinate system x_i to any other system $\bar{x}_a$, where either system is considered as defining the position of an arbitrary point P; the $\bar{x}_a$ are considered either as functions of the x_i or of the original Cartesian system.

The simplest case is that of a function of position ϕ which has a definite value at every point P. It may be considered as a function of the coordinates x_i and is obviously transformed by the relation of "invariance"

$$\bar{\phi}(\bar{x}_1, \bar{x}_2, \bar{x}_3) = \phi(x_1, x_2, x_3).$$

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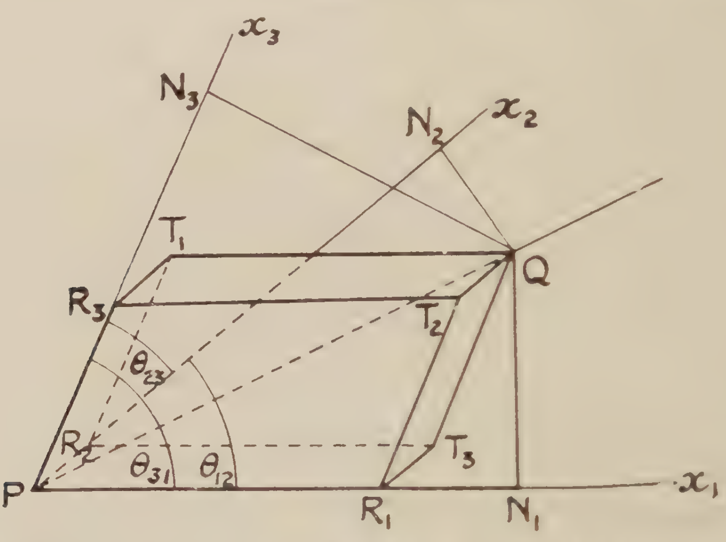
* Blackie and Son, 1927.

† 4th edition, § 328, equation (8).

Before proceeding to discuss the possible ways of expressing a vector function of position it is desirable to remark that the tensor calculus is only concerned with "infinitesimal" vectors. Every vector used in physical problems can in fact be represented by a directed line of infinitesimal length, with the exception of the straight line representing the position of the point P; the position of P is not considered as a vector in the tensor calculus.

We start therefore by considering the infinitesimal vector **A**, which is a function of the position of the point P, as being represented by the line joining P (coordinates x_i) to a neighbouring point Q (coordinates $x_i + dx_i$)*. Consider also the points R_1, R_2, R_3 whose coordinates are $(x_1 + dx_1, x_2, x_3)$ etc. and T_1, T_2, T_3 whose coordinates are $(x_1, x_2 + dx_2, x_3 + dx_3)$ etc. These eight points form to the first order a parallelepiped which is in general oblique angled.



One method of representing the vector **A** is by means of the three components dx^i (dx_1, dx_2, dx_3) which are proportional to the lengths PR_1, PR_2, PR_3. An equally reasonable method would be to make use of quantities proportional to the projections PN_1, PN_2, PN_3 of PQ on the axial lines PR_1, PR_2, PR_3.

Write ds for the length of PQ and consider the formula connecting ds with the dx^i. ds may be considered as an invariant function of **A** and of the position of P. In Cartesian coordinates

$$ds^2 = dX_1{}^2 + dX_2{}^2 + dX_3{}^2 \qquad \text{......(2·1)},$$

which may be written

$$dX_a dX^a*,$$

with

$$dX_a = dX^a.$$

By direct transformation we have

$$ds^2 = \frac{\partial X_a}{\partial x_i} \cdot \frac{\partial X^a}{\partial x_j} dx^i dx^j*,$$

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* See Appendix A.

or writing g_{ij} for $\dfrac{\partial X_a}{\partial x_i}\cdot\dfrac{\partial X^a}{\partial x_j}$ ......(2·2),

$$ds^2 = g_{ij}\,dx^i\,dx^j \qquad\text{......(2·3)},$$

with the relation $g_{ij} = g_{ji}$. By definition g_{ij} are functions of position which depend only on the particular system of coordinates, not on the particular vector **A**.

From the geometry of the parallelepiped, writing θ_{23} for the angle R_2PR_3, etc.,

$$ds^2 = \Sigma PR_1{}^2 + 2\Sigma PR_2PR_3\cos\theta_{23} \qquad\text{......(2·4)}.$$

Comparing equations (2·3) and (2·4), we obtain the equations

$$PR_1 = g_{11}{}^{\frac{1}{2}}\,dx_1 \qquad\text{......(2·5)},$$

etc., and $\cos\theta_{23} = \dfrac{g_{23}}{g_{22}{}^{\frac{1}{2}}\,g_{33}{}^{\frac{1}{2}}}$ ......(2·6),

etc. Equations (2·5) and (2·6) connect the g_{ij} with the geometry of the parallelepiped.

We are now in a position to define two types of components of the vector **A**. These will be described as the "contravariant" components A^a and the "covariant" components A_a. The A^a are defined at once by the three equations

$$A^a = dx^a \qquad\text{......(2·7)},$$

so that equation (2·5) gives $A^1 = PR_1/g_{11}{}^{\frac{1}{2}}$ ......(2·8).

Again PN_1, the projection of PQ on PR_1, is equal to the sum of the projections of PR_1, PR_2, PR_3 on PR_1 so that

$$\begin{aligned}PN_1 &= PR_1 + PR_2.\cos\theta_{12} + PR_3.\cos\theta_{13}\\ &= g_{11}{}^{\frac{1}{2}}\,dx_1 + \frac{g_{12}}{g_{11}{}^{\frac{1}{2}}\,g_{22}{}^{\frac{1}{2}}}\cdot g_{22}{}^{\frac{1}{2}}\,dx_2 + \frac{g_{13}}{g_{11}{}^{\frac{1}{2}}\,g_{33}{}^{\frac{1}{2}}}\cdot g_{33}{}^{\frac{1}{2}}\,dx_3\\ &= \frac{1}{g_{11}{}^{\frac{1}{2}}}\,g_{1a}\,dx_a{}^{*} \qquad\text{......(2·9)}.\end{aligned}$$

We shall define the covariant component A_a of **A** by the equation

$$A_a = g_{ab}\,dx^b \qquad\text{......(2·10)}.$$

It follows from equation (2·9) that

$$A_1 = g_{11}{}^{\frac{1}{2}}\,PN_1 \qquad\text{......(2·11)},$$

etc., which give the geometrical definition of A_a in a form similar to equation (2·8) for A^a. Alternative definitions of A^a and A_a may be given with reference to the parallelepiped consisting of planes perpendicular to PR_1, PR_2, PR_3. It appears that A^a and A_a are equally well qualified to be considered as components of the vector **A**.

When the coordinates are orthogonal the point N_1 coincides with R_1, and PN_1 is the "component" of **A** as usually defined. Also, when $i \neq j$, g_{ij} is zero by equation (2·6). It follows from equations (2·8) and (2·11) that

$$g_{11}{}^{\frac{1}{2}}\,A^1 = PN_1 = \frac{1}{g_{11}{}^{\frac{1}{2}}}\,A_1 \qquad\text{......(2·12)}.$$

In Cartesian coordinates $g_{11} = 1$ and the distinction between A^a and A_a disappears.

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* See Appendix A.

§ 3. SOME FUNDAMENTAL ALGEBRAICAL RESULTS

At this stage it is convenient to define the most general type of tensor and to obtain its laws of transformation, showing that A^i, A_i as defined above are particular cases. We take dx^i as the typical contravariant tensor, which evidently satisfies the law of transformation

$$d\bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^a} dx^a \qquad \text{......(3·1).}$$

An alternative definition of a covariant vector U_i is given by the equation

$$\phi = U_i dx^i \qquad \text{......(3·2).}$$

It may be shown* that the values of U_i in any other system of coordinates are consistently and uniquely defined by the condition that ϕ is invariant for any arbitrary values of dx^i, and that the U_i so defined satisfy the law of transformation

$$\bar{U}_i = \frac{\partial x_a}{\partial \bar{x}_i} U_a \dagger \qquad \text{......(3·3).}$$

Both the laws of transformation (3·1) and (3·3) may be shown to be transitive and also invertible, so that e.g.

$$U_a = \frac{\partial \bar{x}_i}{\partial x_a} \bar{U}_i \qquad \text{......(3·4).}$$

The covariant vector U_i has still to be identified with the covariant component A_i, defined in equation (2·10).

In addition to the contravariant and covariant vectors having three components it is possible to define a more general type of tensor having 9, 27, etc. components by an extension of equation (3·2). For example write

$$\phi = A_{ab}{}^{ijk} U_i U_j' U_k'' dx^a dx'^b \qquad \text{......(3·5),}$$

so that $A_{ab}{}^{ijk}$ has 3^5 components. It may be shown that components $\bar{A}_{ab}{}^{ijk}$ in any other system of coordinates are consistently and uniquely defined by the condition that ϕ is invariant for arbitrary values of the U_i and dx^a, and that the $A_{ab}{}^{ijk}$ satisfy the law of transformation

$$\bar{A}_{ab}{}^{ijk} = \frac{\partial \bar{x}_i}{\partial x_l} \frac{\partial \bar{x}_j}{\partial x_m} \frac{\partial \bar{x}_k}{\partial x_n} \frac{\partial x_f}{\partial \bar{x}_a} \frac{\partial x_g}{\partial \bar{x}_b} A_{fg}{}^{lmn} \qquad \text{......(3·6).}$$

The $A_{ab}{}^{ijk}$ are described as the components of a tensor of the fifth rank having three elements of contravariant and two of covariant character.

As a particular example, a comparison of equation (2·3) with equation (3·5) above, shows that the g_{ij} are components of a covariant tensor of the second rank; it is of fundamental importance, being a function of the coordinate system only.

It may be shown, as a particular case of (3·5), that the "outer product" of any number of vectors or tensors is itself a tensor, e.g.

$$A^i . B_k{}^j = C_k{}^{ij} \qquad \text{......(3·7),}$$

is a tensor of the kind indicated.

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* Levi-Civita, *The Absolute Differential Calculus*, chap. 4, § 11.

† A simple method of remembering this law of transformation is afforded by equation (6·1) below.

Contraction. It may be shown* that the result of putting a lower suffix equal to an upper suffix in any tensor and performing the indicated summation†, is itself a tensor. Thus A_{ka}^{ia} is a tensor obtained from the tensor A_{kl}^{ij} by the process of "contraction." In particular the result of contracting the "mixed" tensor A_j^i is the scalar quantity A_a^a.

A further extension of equation (3·5) is that if (e.g.) B_k^{ij} is an *arbitrary* tensor then the condition that any product (contracted or not) of B_k^{ij} into A_{mn}^l is a tensor of the kind indicated, defines A_{mn}^l uniquely and consistently in any coordinate system such that it is a tensor of the kind indicated.

A very specialised type of mixed tensor is defined as follows:

$$\begin{aligned}\delta_j^i &= 0 \text{ when } i \neq j' \\ &= 1 \text{ when } i = j\end{aligned} \qquad \text{......(3·8).}$$

The fact that δ_j^i satisfies the correct law of transformation may be verified by direct substitution. By virtue of the above rule of contraction it may be verified that the contracted product of δ_j^i into e.g. a contravariant vector A^k is given by

$$\delta_j^i A^j = A^i \qquad \text{......(3·9),}$$

so that δ_j^i has the property of substituting one suffix letter for another in any tensor. Other important properties which may be easily verified are embodied in the equation

$$\delta_j^i = g^{ia} g_{aj} ‡ = \frac{\partial \bar{x}^i}{\partial x^a} \frac{\partial x^a}{\partial \bar{x}^j} \qquad \text{......(3·10).}$$

Application to geometrical vectors. We may now return to the subject of the representation of geometrical vectors by means of the tensor notation. It follows from a remark in the section on contraction, that the covariant component as defined in equation (2·10) is equivalent to the covariant vector as defined in equation (3·2) and satisfies the same law of transformation, equation (3·3). The three simultaneous linear equations

$$A_i = g_{ia} A^a \qquad \text{......(3·11),}$$

may be solved for the A^a in the form

$$A^a = g^{aj} A_j \qquad \text{......(3·12),}$$

where, according to the ordinary rule for solving simultaneous equations, g^{ij} is the cofactor of g_{ij} in the determinant having g_{ij} as elements divided by the value g of the determinant, and it follows from (3·12) that g^{ij} is a tensor. Thus the fundamental tensors g_{ij} and g^{ij} can be used to convert the contravariant components of a vector into the covariant components and *vice versa*.

We are now in a position to show how either the A^i or A_i are convenient for representing the vector **A** in any system of coordinates. The first observation is that the condition $A_i = 0$ involves the vanishing of **A**. Again, equations of the type

$$A_i = B_i,$$

hold in any system of coordinates, and involve the identity of the vectors **A** and **B**.

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* Levi-Civita, *loc. cit.* chap. 4, § 9. † Appendix A. ‡ g^{ia} is defined in the next section.

In view of the linearity of the equations of transformation, it follows that the equation

$$K_i = A_i + B_i + C_i \qquad \text{......(3·13)},$$

holds in any system of coordinates, and it may be shown geometrically that the vector **K** is equal to the sum of the vectors **A**, **B**, **C** according to the parallelogram law.

§ 4. GEOMETRICAL EXAMPLES OF TENSORS OF THE SECOND ORDER

The "outer" product of two vectors. The results of § 3 show that the "outer" product of the components of two vectors **A** and **B**, e.g. $A^i B^j$ is a tensor of the second rank having nine components. Since there are two types of component of each vector, we can construct four tensors of the second rank, $A^i B^j$, $A^i B_j$, $A_i B^j$, $A_i B_j$ (one contravariant, two mixed, and one covariant), which may be considered as being "components" of a tensor (generalised vector) of the second rank.

Linear vector function of a vector. We consider a vector **F** which is a function of the arbitrary vector **A**. **F** (A) is described as a linear function if it satisfies the following conditions: (i) if **A** is fixed in direction, then **F** is fixed in direction, and of magnitude proportional to the magnitude of **A**; (ii) if **A** (1), **A** (2) are any two values of **A**, and **A** (3) is their vector sum, then **F** $\{A\,(3)\}$ is the vector sum of **F** $\{A\,(1)\}$ and **F** $\{A\,(2)\}$. Under these conditions it may be shown that the components of **F** are equal to the reduced product of the components of **A** into a tensor T* of the second rank, e.g.

$$F^i = T^{ij} A_j \qquad \text{......(4·1)}.$$

Proof. Let **A** (1), **A** (2), **A** (3) be three values of **A** having different directions. Substitute these and the corresponding values of **F** in equation (4·1) and we obtain nine simultaneous equations which in virtue of the results of § 3 determine the components of T as a tensor in any system of coordinates. Now any value **A** (4) of **A** may be expressed in the form

$$\mathbf{A}\,(4) = \alpha_1\,\mathbf{A}\,(1) + \alpha_2\,\mathbf{A}\,(2) + \alpha_3\,\mathbf{A}\,(3) \qquad \text{......(4·2)},$$

where α_1, α_2, α_3 are scalar constants and it follows from the conditions satisfied by **F** that

$$\mathbf{F}\,(4) = \alpha_1\,\mathbf{F}\,(1) + \alpha_2\,\mathbf{F}\,(2) + \alpha_3\,\mathbf{F}\,(3) \qquad \text{......(4·3)},$$

and therefore **F** (4) and **A** (4) satisfy equation (4·1). Thus equation (4·1) represents the necessary and sufficient condition that **F** is a linear vector function of **A**. The tensor T is of the most general type, unlike the example of the last section (product of two vectors).

§ 5. CONTRACTION OF TENSORS OF THE SECOND ORDER

From any tensor T* of the second rank we can obtain by processes of contraction a scalar and a vector which may be described as the "scalar contraction" and the "vector contraction" of T respectively.

Scalar contraction. The contracted mixed component T^a_a* of the tensor T is a scalar: the scalar contraction of T.

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* By analogy with the outer product we consider any tensor T, of the second rank, to have the four components: T^{ij}, $T^i{}_j$, $T_i{}^j$, T_{ij}. Any three of these may be expressed in terms of the fourth by relations identical with those obtainable by application of equations (3·11) or (3·12) to the outer product. It follows that $T^a{}_a = T_a{}^a$, and this contraction will therefore be written T^a_a.

Vector contraction. Epsilon tensors. It appears on examination that the antisymmetrical tensor $T^{ab} - T^{ba}$ has only three distinct components. It may be reduced to a vector by means of the fundamental auxiliary tensor of the third rank E_{abc}* (in space of three dimensions) defined by the conditions

$$E_{abc} = 0,$$

unless a, b and c are all different, when

$$E_{abc} = \pm g^{\frac{1}{2}} \qquad \text{......(5·1)},$$

the sign being that of the corresponding term of the fundamental three-row determinant whose value is g. The corresponding contravariant tensor is defined by

$$E^{ijk} = g^{ia} g^{jb} g^{kc} E_{abc} \qquad \text{......(5·2)},$$

and may be shown to satisfy similar conditions to those of equation (1) but with $g^{\frac{1}{2}}$ replaced by $g^{-\frac{1}{2}}$. An important result which may be easily verified is that

$$E^{ijk} E_{iab} = \delta_a^j \delta_b^k - \delta_b^j \delta_a^k \qquad \text{......(5·3)}.$$

Now consider the vector **L** defined by the alternative equations

$$\begin{aligned} L_i &= \tfrac{1}{2} E_{iab} (T^{ab} - T^{ba}) \\ &= E_{iab} T^{ab} && \text{......(5·4)}, \\ L^i &= \tfrac{1}{2} E^{iab} (T_{ab} - T_{ba}) && \text{......(5·5)}, \end{aligned}$$

which may be shown to be all equivalent. Multiplying both sides of (5·4) by E^{ijk} and using (5·3) we have

$$\begin{aligned} E^{ijk} L_i &= (\delta_a^j \delta_b^k - \delta_b^j \delta_a^k)\, T^{ab} \\ &= T^{jk} - T^{kj} && \text{......(5·6)}, \end{aligned}$$

by equation (3·9). By means of the equations (5·5) and (5·6), the vector **L** (the "vector contraction" of T) may be expressed in terms of the tensor $(T^{jk} - T^{kj})$ and *vice versa*.

Some additional identities involving epsilon tensors which will be required below are given in Appendix B.

Scalar and vector products. Examples of these two types of contracted tensor are given by substituting for the tensor T^{ab} the "outer product" $A^a B^b$ of the two vectors **A** and **B**. These are the scalar product ϕ and the vector product **V** of the two vectors defined by the equations

$$\phi = A^a B_a \qquad \text{......(5·7)},$$

and

$$V_i = E_{ijk} A^j B^k \qquad \text{......(5·8)},$$

similar to equation (5·4). It may be shown that these are identical with the scalar and vector products of two vectors as ordinarily defined.

If we take the scalar product of **V** into a third vector **C** it is obvious geometrically that the result is equal to the volume of the parallelepiped having **A**, **B**, **C** as adjacent edges. Alternative expressions for this volume are therefore

$$\begin{aligned} C_k V^k &\equiv E^{ijk} A_i B_j C_k && \text{......(5·9)}, \\ &\equiv \frac{1}{g^{\frac{1}{2}}} \begin{vmatrix} A_1 B_1 C_1 \\ A_2 B_2 C_2 \\ A_3 B_3 C_3 \end{vmatrix}. \end{aligned}$$

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* Levi-Civita, *loc. cit.* chap. 6, § 8. He uses ϵ in place of E.

§6. DIFFERENTIATION OF SCALAR AND VECTOR FUNCTIONS OF POSITION

Derivative of a scalar. The change $\delta\phi$ in a scalar function ϕ due to a small displacement from the point P to the point Q considered as a vector $PQ = \mathbf{B}$ is obviously given by

$$\delta\phi = \phi_Q - \phi_P = \frac{\partial\phi}{\partial x_a}\, dx^a = \frac{\partial\phi}{\partial x^a}\, B^a \qquad \text{......(6·1).}$$

It follows from equation (3·2), that since $\delta\phi$ is invariant, $\partial\phi/\partial x_a$ is the covariant component of a vector. This vector is the gradient of the scalar ϕ as usually defined.

Derivative of a vector. Example of a linear vector function. For an arbitrary vector function of position **A** we have to consider the effect on **A** of a small displacement from the point P to the point Q, where PQ is considered as an arbitrary vector **B** whose contravariant components are dx^a. Let $\delta\mathbf{A}$ be a vector equal to the difference between $\mathbf{A}_Q$ and $\mathbf{A}_P$ or more exactly to the difference between $\mathbf{A}_Q$ and the vector through Q equal and parallel to $\mathbf{A}_P$. Obviously $\delta\mathbf{A}$ is a function of the vector **B** and it may be shown geometrically that it is a "linear vector function." Hence it is possible to define the four components $A^{i,a}$; $A^{i}_{,a}$; $A_{i}^{,a}$; $A_{i,a}$ of the second rank "tensor derivative"* of **A** by equations such as

$$\delta A^i = A^i_{,a} B^a = A^i_{,a}\, dx^a \qquad \text{......(6·2).}$$

In general when the coordinates are curvilinear $A^i_{,a}$ is not equal to $\partial A^i/\partial x^a$ nor is $\partial A^i/\partial x^a$ a tensor. The following expressions for $A^i_{,a}$ and $A_{i,a}$ are obtained in Appendix C, by direct transformation from Cartesian coordinates:

$$A^i_{,a} = \frac{\partial A^i}{\partial x_a} + \Gamma_{ab}{}^{i} A^b \qquad \text{......(6·3),}$$

$$A_{i,a} = \frac{\partial A_i}{\partial x_a} - \Gamma_{ia}{}^{b} A_b \qquad \text{......(6·4),}$$

where

$$\Gamma_{jk}{}^{i} = \tfrac{1}{2} g^{ia} \left\{ \frac{\partial g_{ja}}{\partial x_k} + \frac{\partial g_{ka}}{\partial x_j} - \frac{\partial g_{jk}}{\partial x_a} \right\} \qquad \text{......(6·5),}$$

is known as a Christoffel symbol of the second kind.

Derivative of a tensor. Similar expressions may be obtained for the derivative of a tensor of any rank, e.g. the tensor A_{ab} has the covariant derivative

$$A_{ab,c} = \frac{\partial A_{ab}}{\partial X_c} - \Gamma_{bc}{}^{i} A_{ai} - \Gamma_{ac}{}^{i} A_{ib} \qquad \text{......(6·6).}$$

It may be shown that the covariant derivatives of the fundamental tensors g_{ij}, g^{ij}, E_{abc} are all zero so that these tensors behave as constants for the purpose of covariant differentiation. It follows that $A_{i,a} = g_{ib} A^b_{,a}$.

Divergence of a vector. The "scalar contraction" of the tensor derivative of A is $A^a_{,a}$ which is identical with the divergence as ordinarily defined. It may be verified that $A^a{}_{,a}$ may be expressed in the following form which does not contain Christoffel symbols:

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* The "comma" notation for the tensor derivative is used throughout the paper. Of the four components of the tensor derivative of a vector (see p. 270, footnote), $A^i_{,a}$ and $A_{i,a}$ are defined by equations (6·3) and (6·4), and satisfy the relation $A_{i,a} = g_{ib} A^b_{,a}$ while $A^{i,a}$ and $A_i^{,a}$ are defined by the relations $A^{i,a} = g^{ab} A^i_{,b}$ and $A_i^{,a} = g^{ab} A_{i,b}$.

$$\text{div.}\,\mathbf{A} \equiv A^a{}_{;a} = g^{-\frac{1}{2}} \frac{\partial}{\partial x_a} (g^{\frac{1}{2}} A^a)^* \qquad \ldots\ldots(6{\cdot}7).$$

Curl of a vector. The "vector contraction" of the tensor derivative of **A** (with sign reversed) is identical with the curl of **A** as ordinarily defined and may be written "curl **A**." From equation (5·5) we have

$$\begin{aligned}(\text{curl}\,A)^i &= E^{iab} A_{b,a} \\ &= \tfrac{1}{2} E^{iab} (A_{b,a} - A_{a,b}) \qquad \ldots\ldots(6{\cdot}8).\end{aligned}$$

When the second form is written out in full the Christoffel symbols cancel and we have therefore

$$(\text{curl}\,A)^i = E^{iab} \frac{\partial A_b}{\partial x_a} \qquad \ldots\ldots(6{\cdot}9),$$

a form independent of Christoffel symbols. Since the operation of obtaining the curl changes a vector into a vector it may be repeated any number of times. For this purpose it is convenient to obtain formulae connecting corresponding components of **A** and curl **A**; these are:

$$(\text{curl}\,A)_i = g_{ia} E^{abc} \frac{\partial A_c}{\partial x_b} \qquad \ldots\ldots(6{\cdot}10),$$

$$(\text{curl}\,A)^i = E^{iab} \frac{\partial}{\partial x_a} (g_{bc} A^c) \qquad \ldots\ldots(6{\cdot}11).$$

Second derivative. By substituting the first derivative of a vector (equation 6·4) in the formula for the first derivative of the corresponding tensor (equation 6·6) we obtain the second derivative of the vector **A** of which e.g. the covariant component may be written $A_{i,a,b}$. It is important to notice that in Euclidean space with which alone we are concerned, the order of covariant differentiation is indifferent; this fact involves a series of complicated identities among the g_s and their derivatives.

Contracted second derivative ∇^2. The contracted second derivative (contravariant component, $A^{i,a}{}_{,a}$) is the component of a vector which is identical with ∇^2**A**. This vector may be reduced to a form independent of Christoffel symbols by using the identity usually written

$$\nabla^2\mathbf{A} \equiv \text{grad. div.}\,\mathbf{A} - \text{curl curl}\,\mathbf{A} \qquad \ldots\ldots(6{\cdot}12),$$

since the divergence, curl, and gradient have all been simplified in this way. A proof of this identity by tensor notation is given in Appendix D and an alternative method of simplifying ∇^2**A** in Appendix E.

Reduced second derivative of a scalar $\left(\frac{\partial\phi}{\partial x^a}\right)^{,a}$†. This is identical with the divergence of the gradient of ϕ and is also usually written $\nabla^2\phi$, but it is important to notice that the formulae for $\nabla^2\phi$ and ∇^2**A** only agree in the special case of Cartesian coordinates.

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* Levi-Civita, *loc. cit.* chap. 6, § 7.

† See footnote, p. 272.

§7. VECTORS AND TENSORS REQUIRED IN HYDRODYNAMICS

Velocity. If a particle of fluid at a point P is displaced to the point $\underset{\sim}{Q}$ in a small time interval δt, then $\underset{\sim}{PQ}$ is an infinitesimal vector and the velocity $\mathbf{u}$ may be defined as the vector

$$\mathbf{u} = \frac{1}{\delta t}\underset{\sim}{PQ} \qquad \text{......(7·1)},$$

the time δt being considered as an independent variable unaffected by change of coordinates.

Acceleration. We shall adopt the Eulerian convention by which the velocity $\mathbf{u}$ at a fixed point P in space is considered as a function of the time and of the coordinates of P, and shall deal only with the case of fixed axes. In determining the acceleration of a given particle of fluid write $\mathbf{u}_P(t)$ and $\mathbf{u}_P(t + \delta t)$ for the velocities at the point P at the beginning and end of the small interval δt. At the end of the interval the particle of fluid will have moved to the point $\underset{\sim}{Q}$ where $\underset{\sim}{PQ} = \mathbf{u}_P\delta t$ to the first order and its velocity will then be $\mathbf{u}_Q(t + \delta t)$. Let $\mathbf{f}$ be the vector acceleration of the particle, then

$$\begin{aligned} \mathbf{f}\delta t &= \mathbf{u}_Q(t+\delta t) - \mathbf{u}_P(t) \\ &= \mathbf{u}_Q(t) - \mathbf{u}_P(t) + \mathbf{u}_P(t+\delta t) - \mathbf{u}_P(t) \end{aligned} \qquad \text{......(7·2)},$$

to the first order; in tensor notation (contravariant components) this becomes:

$$f^i\delta t = \delta u^i + \frac{\partial u^i}{\partial t}\delta t \qquad \text{......(7·3)}.$$

By the results of section 6, δu^i the contravariant component of the difference between the simultaneous velocities at $\underset{\sim}{Q}$ and P is given by

$$\delta u^i = u^i{}_{,a}\,\delta x^a \qquad \text{......(7·4)},$$

where δx^a the contravariant component of the vector $\underset{\sim}{PQ}$ is given by

$$\delta x^a = u^a\delta t \qquad \text{......(7·5)}.$$

Hence we have

$$f^i = u^i{}_{,a}u^a + \frac{\partial u^i}{\partial t} \qquad \text{......(7·6)},$$

and similarly

$$f_i = u_{i,a}u^a + \frac{\partial u_i}{\partial t} \qquad \text{......(7·7)}.$$

A similar formula applied to any vector function of position corresponds to the usual formula for differentiation following the motion of the fluid*.

When $u_{i,a}u^a$ is written out in full it is found that two of the three terms in the Christoffel symbols cancel, leading to the result:

$$u_{i,a}u^a = u^a\frac{\partial u_i}{\partial x_a} - \tfrac{1}{2}u^a u^b\frac{\partial g_{ab}}{\partial x_i} \qquad \text{......(7·8)}.$$

Rate of strain. It is usual in works on hydrodynamics to separate the derivative of the velocity $u_{i,j}$ into two components A_{ij} and B_{ij} given by

$$A_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \qquad \text{......(7·9)},$$

$$B_{ij} = \tfrac{1}{2}(u_{i,j} - u_{j,i}) \qquad \text{......(7·10)},$$

so that

$$u_{ij} = A_{ij} + B_{ij} \qquad \text{......(7·11)}.$$

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* Lamb, *Hydrodynamics*, § 5.

appears from equation (6·8) that the tensor B_{ij} can be expressed in terms of a ector ξ (the curl of the velocity) given by the equation

$$\xi^i = (\text{curl } u)^i = E^{iab} u_{b,a} \qquad \text{......(7·12).}$$

his vector is the "vorticity" as ordinarily defined; the tensor A_{ij} is described as e "rate of strain." The proof given below shows that the stress components are nctions of the rate of strain and independent of the rotation so that it becomes nnecessary to make the separation at this stage. The tensor B or the vector ξ can e expressed in a form independent of Christoffel symbols by means of equations ·10) or (6·11).

For the tensor A it will be found that when A^{ij} is written out in full some of e derivatives of the g_s in the Christoffel symbols cancel, giving the result

$$\begin{aligned} 2A^{ij} &= u^{i,j} + u^{j,i} \\ &= g^{ja}\frac{\partial u^i}{\partial x_a} + g^{ia}\frac{\partial u^j}{\partial x_a} + u^a g^{ib} g^{jc}\frac{\partial g_{bc}}{\partial x_a} \end{aligned} \qquad \text{......(7·13).}$$

§ 8. STRESS

Consider an arbitrary small element of area dS considered as a vector defined s being perpendicular to the plane of the element of area, and equal to it in magniude, and let dF be the resultant force of reaction across this area in the fluid. dF is function of the vector dS and it will be shown that it is a linear vector function as efined in § 4. Consider the equilibrium of the element of fluid bounded by any mall tetrahedron. The forces dF on the four sides have zero resultant to the first rder since the inertia and external forces are of a higher order of small quantities; he vector elements of area of the four sides have zero resultant by the geometry f the tetrahedron. Again, when dS is fixed in direction dF is fixed in direction and roportional to dS. Hence dF is a linear vector function of dS, and by § 4 a tensor of the second rank may be defined by an equation such as

$$dF^i = P^{ij} dS_j \qquad \text{......(8·1).}$$

he stress tensor P then determines dF for all values of dS.

The scalar contraction of P is P^a_a which may be written

$$P^a_a = \frac{dF_1}{dS_1} + \frac{dF_2}{dS_2} + \frac{dF_3}{dS_3} = \frac{dF^1}{dS^1} + \frac{dF^2}{dS^2} + \frac{dF^3}{dS^3} \qquad \text{......(8·2).}$$

t is usual to write $-\frac{1}{3}P^a_a = p$ the "mean pressure" in the fluid.

Equation of moments. Symmetry of P^{ij}. An additional relation may be obtained rom the equation of moments on the fluid contained in an elementary parallelepiped hose adjacent edges are the vectors $d\boldsymbol{\xi}.d\boldsymbol{\eta}.d\boldsymbol{\zeta}$. If dF (ξ) is the stress across the ace bounded by $d\eta$, $d\xi$, the sum of the moments of the forces on the opposite aces about the centre of the parallelepiped is the vector **M** (ξ) equal to the vector roduct of dF (ξ) by $d\boldsymbol{\xi}$, so that

$$M_i(\xi) = E_{ija} dF^j(\xi)\, d\xi^a \qquad \text{......(8·3).}$$

Again the area dS(ξ) of the face of the parallelepiped bounded by $d\eta$ and $d\zeta$ is given by

$$dS_i(\xi) = E_{ibc}\, d\eta^b d\zeta^c \qquad \text{......(8·4).}$$

Adding the moments of the forces on the other pairs of opposite faces, we have for the whole moment

$$\begin{aligned} M_i &= M_i(\xi) + M_i(\eta) + M_i(\zeta) \\ &= E_{ija} P^{jk} dS_k d\xi^a + \dots \qquad \text{......(8·5),} \\ &= \{E_{ija}E_{kbc} + E_{ijb}E_{kca} + E_{ijc}E_{kab}\} P^{jk} d\xi^a d\eta^b d\zeta^c \\ &= \tfrac{1}{2} E_{ijk} (P^{jk} - P^{kj}) E_{abc} d\xi^a d\eta^b d\zeta^c \qquad \text{......(8·6),} \end{aligned}$$

using the result of Appendix B, equation (B·2). Since all other forces are of a higher order of small quantities, it follows that **M** vanishes and hence from equation (8·6)

$$P^{jk} = P^{kj} \qquad \text{......(8·7).}$$

Equation of motion in terms of stress. It is now possible to write down the equations of motion of the fluid in terms of the stress tensor, by application of d'Alembert's principle to the equilibrium of the fluid contained within the elementary parallelepiped specified in the last paragraph. The difference $\delta d\mathbf{F}(\xi)$ between the stresses on the opposite faces bounded by $d\eta$, $d\zeta$ is given by

$$\begin{aligned} \delta dF(\xi)^i &= dF(\xi)^i{}_{,a} d\xi^a \\ &= P^{ij}{}_{,a} dS(\xi)_j d\xi^a \\ &= P^{ij}{}_{,a} E_{jbc} d\xi^a d\eta^b d\zeta^c \qquad \text{......(8·8).} \end{aligned}$$

Again, the mass acceleration of the element of fluid is

$$\rho f^i E_{abc} d\xi^a d\eta^b d\zeta^c$$

using the expression for the element of volume given in equation (5·9). Equating this to the resultant stress (assuming no external forces), we have the equations

$$\begin{aligned} \rho f^i E_{abc} d\xi^a d\eta^b d\zeta^c &= \delta dF(\xi)^i + \delta dF(\eta)^i + \delta dF(\zeta)^i \\ &= \{P^{ij}{}_{,a} E_{jbc} + P^{ij}{}_{,b} E_{jca} + P^{ij}{}_{,c} E_{jab}\} d\xi^a d\eta^b d\zeta^c \\ &= P^{ia}{}_{,a} E_{abc} d\xi^a d\eta^b d\zeta^c \qquad \text{......(8·9),} \end{aligned}$$

using the result of equation (B·1). Hence

$$\begin{aligned} P^{ia}{}_{,a} &= \rho f \\ &= \rho \left\{ \frac{\partial u^i}{\partial t} + u^i{}_{,a} u^a \right\} \qquad \text{......(8·10),} \end{aligned}$$

which is the required equation of motion.

§9. PROOF OF THE EQUATIONS OF VISCOUS FLOW

In order to obtain the final equations of motion in terms of the velocities it remains to formulate a relation between the stress components and the velocities. Consider the vector $d\mathbf{F}$ representing the stress across an arbitrary element of area $d\mathbf{S}$ in the fluid. The vector $d\mathbf{F}$ must satisfy the following conditions: (i) It must be a linear vector function of $d\mathbf{S}$. (ii) It must be a linear function of the first tensor derivative of the velocity vector $\mathbf{u}$. This corresponds to the ordinary assumption on which the theory of viscous flow is based. (iii) The function must contain no additional variables other than scalar functions of the density, temperature, etc. since the properties of the fluid are non-directional.

There is no need at this stage to make use of the additional condition that the results must be independent of the vorticity; it will appear that this follows automatically from the condition proved in equation (8·7) that the stress tensor P is symmetrical.

The vector $d\mathbf{F}$ must therefore be given by an equation of the form

$$d\mathbf{F} = \alpha d\mathbf{A} + \beta d\mathbf{B} + \gamma d\mathbf{C} + \dots \qquad \dots\dots(9\cdot1),$$

here $d\mathbf{A}$, $d\mathbf{B}$, $d\mathbf{C}$, etc. are vector functions of $d\mathbf{S}$ and the tensor derivative of $\mathbf{u}$ hich satisfy the above conditions, and α, β, γ, etc. are scalars.

It appears on examination that there are only four independent vectors satis-ing the above conditions. These are

$$d\mathbf{A} = d\mathbf{S} \qquad \dots\dots(9\cdot2),$$

$$d\mathbf{A}' = u^a_{\ ,a}\, d\mathbf{S},$$

$d\mathbf{B}$ = the change of $\mathbf{u}$ in a displacement PQ equal to the vector $d\mathbf{S}$ (i.e. perpendicular to $d\mathbf{S}$ and equal to it). Hence from equation (6·2)

$$dB^i = dS^a u^i_{\ ,a} \qquad \dots\dots(9\cdot3),$$

$d\mathbf{C}$ = the gradient of the scalar product of $\mathbf{u}$ and $d\mathbf{S}$ treating $d\mathbf{S}$ as a constant.

$$dC_i = \frac{\partial}{\partial x_i}(dS_a u^a), \ (dS_a \text{ constant}),$$
$$= dS_a u^a_{\ ,i} \qquad \dots\dots(9\cdot4).$$

ence we may write

$$dF^i = \alpha dS^i + \beta dS^a u^i_{\ ,a} + \gamma dS_a u^{a,i}$$
$$= \{\alpha g^{ia} + \beta u^{i,a} + \gamma u^{a,i}\}\, dS_a \qquad \dots\dots(9\cdot5),$$

here

$$\alpha = \alpha_0 + \alpha_1 u^a_{\ ,a} \qquad \dots\dots(9\cdot6),$$

nd $\alpha_0, \alpha_1, \beta, \gamma$ are scalar functions of the temperature, density etc. only. It is nteresting to notice that the last term will contribute nothing to the equations of otion for an incompressible fluid. Comparing with equation (8·1), it follows, ince $d\mathbf{S}$ is arbitrary, that

$$P^{ij} = \alpha g^{ij} + \beta u^{i,j} + \gamma u^{j,i}$$
$$= \alpha g^{ij} + \beta(u^{i,j} + u^{j,i}) \qquad \dots\dots(9\cdot7),$$

being set equal to β since P^{ij} has been proved to be symmetrical. This proves that P^{ij} involves the pure rate of strain only and is independent of the vorticity.

We might equally well start with equation (9·7), basing it on the assumption hat P^{ij} is a linear function of the pure rate of strain. The above development, eing based on equations between vectors, appears, however, to be susceptible of more definite physical interpretation. Thus for an arbitrary element of area, the irst term $\alpha d\mathbf{S}$ represents a normal pressure on the element; the second term $\beta d\mathbf{B}$ epresents a force parallel and proportional to the rate of change of velocity in a irection normal to $d\mathbf{S}$; the third term $\gamma d\mathbf{C}$ is required to maintain the rotational quilibrium of a volume element of fluid.

We can now write down the equations of motion by substituting the values of P^{ij} from equation (9·7) in equation (8·10). The resulting equation is

$$\rho f^i = P^{ia}_{\ \ ,a}$$
$$= g^{ia}\frac{\partial \alpha}{\partial x_a} + \beta\{u^{i,a}_{\ \ ,a} + u^{a,i}_{\ \ ,a}\}$$
$$= g^{ia}\frac{\partial}{\partial x_a}\{\alpha + \beta u^b_{\ ,b}\} + \beta u^{i,a}_{\ \ ,a} \qquad \dots\dots(9\cdot8).$$

It is here assumed that β is independent of position, and in reducing the last term use is made of the condition that the order of differentiation is indifferent.

It is usual to express α in terms of the mean pressure $p\ (= -\tfrac{1}{3}P^a_a)$ defined by equation (8·2) above. Equation (9·7) may be written

$$P^i_j = \alpha\delta^i_j + \beta\,(u^i{}_{,j} + u_j{}^{,i}) \qquad \text{......(9·9).}$$

Contracting by putting $i = j$ gives

$$-\,3p \equiv P^a_a = 3\alpha + 2\beta u^a{}_{,a}$$

or

$$\alpha = -\,p - \tfrac{2}{3}\beta u^a{}_{,a} \qquad \text{......(9·10).}$$

Substitution in (9·8) (f^i being written in full and β identified with μ, the coefficient of viscosity) gives*

$$\rho\left\{\frac{\partial u^i}{\partial t} + u^i{}_{,a}u^a\right\} = g^{ia}\frac{\partial}{\partial x_a}\{-\,p + \tfrac{1}{3}\mu u^a{}_{,a}\} + \mu u^{i,a}{}_{,a} \qquad \text{(9·11),}$$

as the final form of the equations of motion.

The corresponding expression for P^{ij} is

$$P^{ij} = -\,g^{ij}\,(p + \tfrac{2}{3}\,\mu u^a{}_{,a}) + \mu\,(u^{i,j} + u^{j,i}) \qquad \text{......(9·12).}$$

It is important to notice that in virtue of equations (9·10) and (9·6), p may be expressed in the form

$$-\,p = \alpha_0 + (\alpha_1 + \tfrac{2}{3}\mu)\,u^a{}_{,a}$$

where α_0, α_1 are functions of density and temperature only†.

The equations of motion for covariant components and the expression for mixed and covariant components of the stress tensor can be written down at once from equations (9·11) and (9·12).

§ 10. REDUCTION TO GENERAL ORTHOGONAL COORDINATES

It has been shown that the condition that the coordinate system is orthogonal is that g_{ij} is zero when $i \neq j$. It is convenient to write

$$ds^2 = a^2\,dx^2 + b^2\,dy^2 + c^2dz^2 \qquad \text{......(10·1),}$$

so that

$$g_{11} = a^2,\ \text{etc.,}\quad g = a^2\,b^2\,c^2,\quad g^{11} = 1/a^2\ \text{etc.} \qquad \text{......(10·2).}$$

We shall also make use of the ordinary components of velocity, which will be denoted by U, V, W. It was shown in equation (2·12) that they satisfy the relations

$$u^1 = \sqrt{g^{11}}U = U/a,\ \text{etc.} \qquad \text{......(10·3),}$$

and

$$u_1 = \sqrt{g_{11}}U = aU,\ \text{etc.} \qquad \text{......(10·4).}$$

Again if $dS\,(1)$, $dS\,(2)$, $dS\,(3)$ are actual elements of area of the three coordinate surfaces, the three ordinary components of stress across $dS\,(1)$ are written in the ordinary notation $p_{11}\,dS\,(1)$, $p_{12}\,dS\,(1)$, $p_{13}\,dS\,(1)$. In orthogonal coordinates the covariant components of $dS\,(1)$ are $\{adS\,(1),\,0,\,0\}$ and the contravariant components of the stress are:

$$\{(1/a)\,p_{11}dS\,(1),\quad (1/b)\,p_{12}dS\,(1),\quad (1/C)p_{13}dS\,(1)\}$$

by equation (2·12). Hence equation (8·1) gives:

$$\left.\begin{aligned}(1/a)p_{11}dS\,(1) &= aP^{11}dS\,(1)\\ (1/b)\,p_{12}dS\,(1) &= aP^{12}dS\,(1)\end{aligned}\right\} \qquad \text{......(10·5),}$$

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* The equation was given in this form by Milne, in *Proc. Camb. Phil. Soc.* **20**, 344 (1921).

† See Lamb, *Hydrodynamics*, 4th edition, § 358, equation (1), where μ' is written in place of $\alpha_1 + \tfrac{2}{3}\mu$.

c., so that the nine components of stress as usually defined are

$$\left.\begin{matrix} a^2P^{11}, & abP^{12}, & acP^{13} \\ abP^{21}, & b^2P^{22}, & bcP^{23} \\ acP^{31}, & bcP^{32}, & c^2P^{33} \end{matrix}\right\} \qquad \text{......(10·6).}$$

All the expressions occurring in the equations of motion and in the expressions r the stresses have been reduced to forms which do not involve Christoffel mbols*; it is only necessary to substitute the above values to obtain the equations f motion in orthogonal curvilinear coordinates in full.

For the equations of motion in the ordinary components it is sufficient to ultiply the first contravariant equation by a or divide the covariant equation by a; r the stresses, the above expression in terms of the contravariant components or e corresponding expressions in terms of the mixed or covariant components ay be used. It is sufficient to work out a single example in full and we shall hoose the "curl" which is required in reducing $u^{i,a}{}_{,a}$ by using the formula

$$u^{i,a}{}_{,a} = g^{ia}\frac{\partial}{\partial x_a}(u^a{}_{,a}) - (\text{curl curl } u)^i \qquad \text{......(10·7).}$$

Ve have from equation (6·11)

$$(\text{curl } u)^i = E^{ijk}\frac{\partial}{\partial x_j}(g_{kl}\,u^l).$$

ence for the first component in orthogonal coordinates

$$(\text{curl } u)^1 = E^{123}\frac{\partial}{\partial x_2}g_{33}u^3 + E^{132}\frac{\partial}{\partial x_3}g_{22}u^2$$

ince the summation for the g_s disappears) or in the notation of this section,

First component of curl $(U, V, W) = a\,(\text{curl } u)^1$

$$= \frac{a}{abc}\left\{\frac{\partial}{\partial y}c^2.\frac{W}{c} - \frac{\partial}{\partial z}b^2.\frac{V}{b}\right\}.$$

The first of the three equations of viscous flow in general orthogonal coordinates ay now be obtained by multiplying the first contravariant equation by a; the esulting equation is:

$$\rho\left\{\frac{\partial U}{\partial t} + \frac{1}{a}\left(\frac{U}{a}\frac{\partial}{\partial x} + \frac{V}{b}\frac{\partial}{\partial y} + \frac{W}{c}\frac{\partial}{\partial z}\right)aU - \frac{U^2}{a^2}\frac{\partial a}{\partial x} - \frac{V^2}{ab}\frac{\partial b}{\partial x} - \frac{W^2}{ac}\frac{\partial c}{\partial x}\right\}$$
$$= -\frac{1}{a}\frac{\partial p}{\partial x} + \frac{4\mu}{3a}\frac{\partial}{\partial x}\frac{1}{abc}\left(\frac{\partial bcU}{\partial x} + \frac{\partial caV}{\partial y} + \frac{\partial abW}{\partial z}\right)$$
$$- \frac{\mu}{bc}\left\{\frac{\partial}{\partial y}\frac{c}{ab}\left(\frac{\partial bV}{\partial x} - \frac{\partial aU}{\partial y}\right) - \frac{\partial}{\partial z}\cdot\frac{b}{ac}\left(\frac{\partial aU}{\partial z} - \frac{\partial cW}{\partial x}\right)\right\} \qquad \text{......(10·8).}$$

The typical equations for the stresses may similarly be derived from equations 7·13) and 9·12) in the form

$$p_{11} = -p - \frac{2\mu}{3}\frac{1}{abc}\left(\frac{\partial bcU}{\partial x} + \frac{\partial caV}{\partial y} + \frac{\partial abW}{\partial z}\right) + \frac{2\mu}{a}\left\{\frac{\partial U}{\partial x} + \frac{V}{b}\frac{\partial a}{\partial y} + \frac{W}{c}\frac{\partial a}{\partial z}\right\} \qquad \text{...(10·9),}$$

$$p_{23} = \mu\left\{\frac{b}{c}\frac{\partial}{\partial z}\left(\frac{V}{b}\right) + \frac{c}{b}\frac{\partial}{\partial y}\left(\frac{W}{c}\right)\right\} \qquad \text{......(10·10).}$$

* $u_{i,a}u^a$ in equation (7·8); $u^a{}_{,a}$ in equation (6·7); $u^{i,j} + u^{j,i}$ in equation (7·13); $u^{i,a}{}_{,a}$ in equations 6·12), (6·7) and (6·11), or Appendix E.

The equations for any special case of orthogonal coordinates such as spherical polar coordinates may be obtained at once from the expression for ds^2

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$

as compared with

$$ds^2 = a^2 dx^2 + b^2 dy^2 + c^2 dz^2,$$

and it is only necessary to substitute

$$x = r,\ y = \theta,\ z = \phi,\ a = 1,\ b = r,\ c = r \sin \theta,\ U = U_r,\ V = U_\theta,\ W = U_\phi$$

to get the first equation, and the proper permutations of a, b, c, U, V, W to get the second and third equations.

§ 11. ELIMINATION OF THE PRESSURES

By the taking of the curl of both sides of the equations of motion (equation 9·11) the pressure p is eliminated by virtue of the obvious result that the curl of the gradient of any scalar is zero. The resulting contravariant equation is

$$\frac{\partial}{\partial t}(\text{curl}\, u^j)^i + \{\text{curl}\,(u^j{}_{,b} u^b)\}^i = \nu\, (\text{curl}\, u^{i,a}{}_{,a})^{i*} \qquad \text{......(11·1)},$$

where ν is written for μ/ρ the kinematic coefficient of viscosity. Using the relation

$$u^{i,a}{}_{,a} = g^{ib} \frac{\partial}{\partial x_b} (u^a{}_{,a}) - (\text{curl curl}\, u)^i \qquad \text{......(11·2)},$$

the last term reduces to

$$-\nu\, (\text{curl curl curl}\, u)^i.$$

The details of the reduction of the second term are given in Appendix F. Writing ξ for curl u, as in equation (7·12), we can show that

$$\{\text{curl}\,(u^j{}_{,b} u^b)\}^i = u^a \frac{\partial \xi^i}{\partial x_a} - \xi^a \frac{\partial u^i}{\partial x_a} + (u^a{}_{,a})\, \xi^i \qquad \text{......(11·3)}.$$

The equation of motion therefore reduces to

$$\frac{\partial}{\partial t} \xi^i + u^a \frac{\partial \xi^i}{\partial x_a} - \xi^a \frac{\partial u^i}{\partial x_a} + (u^a{}_{,a})\, \xi^i = -\nu\, (\text{curl curl}\, \xi)^i \qquad \text{...(11·4)}\dagger,$$

with

$$\boldsymbol{\xi} = \text{curl}\, \mathbf{u}.$$

For an incompressible fluid it is possible to define a "vector potential" **h**‡ by the relations

$$\mathbf{u} = \text{curl}\, \mathbf{h}$$

$$h^a{}_{,a} = 0 \qquad \text{......(11·5)},$$

so that

$$\boldsymbol{\xi} = \text{curl curl}\, \mathbf{h} \qquad \text{......(11·6)},$$

and the last term on the left hand side of equation (11·4) vanishes.

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* The notation (curl A^i)^i stands for a contravariant component of the curl of the vector whose contravariant components are A^i, and is identical with (curl A)^i. The former notation is necessary when A^i is an expression such as $u^i{}_{,b} u^b$.

† Lamb, *Hydrodynamics*, 4th edition, § 328, equation (8).

‡ *Ibid.* § 148.

§ 12. MOTION SYMMETRICAL ABOUT AN AXIS

The last result leads directly to Stokes' form for the equation of viscous flow when the motion is symmetrical about an axis. We take for the third coordinate x_3, the angle of azimuth of the point P with reference to the axis of symmetry, and for x_1, x_2 any two-dimensional system of coordinates in planes through the axis. It follows that

$$ds^2 = g_{11}dx^2_1 + 2g_{12}dx_1dx_2 + g_{22}dx_2{}^2 + \varpi^2 dx_3{}^2,$$

and that all the g_s are independent of x_3, ϖ being the normal distance of P from the axis.

We now assume for the components of the vector potential **h** that $h_1 = h_2 = 0$; then it may be verified that $u_3 = 0$ and that u_1, u_2 are functions of h_3 which automatically satisfy the equation of continuity for an incompressible fluid in virtue of the identity

$$\text{div curl } \mathbf{A} \equiv 0$$

and therefore the above assumption with regard to **h** gives the most general motion of an incompressible fluid of the kind considered. It may also be shown for

$$(\text{curl curl } \mathbf{h}) \equiv \xi$$

that
$$\xi_1 = \xi_2 = 0,$$

and similarly that
$$(\text{curl curl } \xi)_1 = (\text{curl curl } \xi)_2 = 0,$$

all variables being independent of x_3. On examination it appears that the equations of motion in the form (11·4) reduce to a single equation in the variable h_3.

In order to identify h_3 with Stokes' stream function ψ, we proceed to obtain an expression for the flow across any surface of revolution bounded by a circle about the axis. By Stokes' theorem and the definition of the vector potential **h** for an incompressible fluid, this is equal to the line integral of **h** taken round the curve, i.e. to $\int h_a dx^a$ taken round the curve. In the present case this reduces to

$$\int_0^{2\pi} h_3 dx_3 = 2\pi h_3$$

which agrees with the definition of Stokes' stream function ψ*.

It is possible to reduce the equations of motion in the form (11·4) to a single tensor equation in the variables $x_1, x_2 : h_3\ (=\psi)$, ϖ and ξ_3 being treated as scalars. It is sufficient for the present purpose to consider the special case in which the two-dimensional coordinates are orthogonal.

Orthogonal coordinates symmetrical round an axis. Using the notation of § 10, putting $c = \varpi$, and using equation (6·10), we can verify that

$$u_1 = (\text{curl } h)_1 = \frac{a}{\varpi b}\frac{\partial\psi}{\partial y} \qquad \text{......(12·1)},$$

$$u_2 = (\text{curl } h)_2 = -\frac{b}{\varpi a}\frac{\partial\psi}{\partial x} \qquad \text{......(12·2)},$$

$$u_3 = 0,$$

---

* Lamb, *Hydrodynamics*, § 94.

writing ψ in place of h_3. Hence

$$U = \frac{1}{\varpi b}\frac{\partial \psi}{\partial y} \qquad \text{......(12·3)},$$

$$V = -\frac{1}{\varpi a}\frac{\partial \psi}{\partial x} \qquad \text{......(12·4)}.$$

Further, we have

$$\xi_1 = \xi_2 = 0,$$

$$\begin{aligned} \xi_3 &= (\text{curl curl } h)_3 \\ &= -\frac{\varpi}{ab}\left\{\frac{\partial}{\partial x}\frac{b}{\varpi a}\frac{\partial \psi}{\partial x} + \frac{\partial}{\partial y}\frac{a}{\varpi b}\frac{\partial \psi}{\partial y}\right\} \qquad \text{......(12·5)}, \\ &= -D(\psi), \end{aligned}$$

say. It is of interest to notice how this expression differs from the expression for $\nabla^2\phi$ in two dimensions, where ϕ is a scalar; this takes the form

$$\nabla^2\phi = \frac{1}{ab}\left\{\frac{\partial}{\partial x}\frac{b}{a}\frac{\partial \phi}{\partial x} + \frac{\partial}{\partial y}\frac{a}{b}\frac{\partial \phi}{\partial y}\right\} \qquad \text{......(12·6)}.$$

The equations of motion (11·4) may now be written down in the form

$$\frac{\partial \xi_3}{\partial t} + \frac{\varpi}{ab}\left\{\frac{\partial \psi}{\partial y}\cdot\frac{\partial}{\partial x}\left(\frac{\xi_3}{\varpi^2}\right) - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}\left(\frac{\xi_3}{\varpi^2}\right)\right\} = -\nu D(\xi_3) \qquad \text{......(12·7)},$$

subject to equation (12·5) above. The vorticity ω as usually defined satisfies the relation

$$\omega = \xi_3/\varpi.$$

APPENDIXES

Appendix A. *Summary of notation and conventions.*

A tensor is expressed by the notation

$$A_c{}^{iab}$$

in which there may be any number of suffixes. The upper or lower position of the suffix is important as denoting contravariance or covariance. If the suffixes are all different it is understood that a suffix may take any of the values 1, 2, 3 (in three dimensions); the above expression therefore stands for 3^4 different components. The only case in which the suffix may be repeated is when there are two identical suffixes, one upper and one lower. By the *summation convention* an expression such as

$$A_{ab}{}^{iab}$$

stands for $\sum_{a=1,2,3}\sum_{b=1,2,3} A_{ab}{}^{iab}$, the signs of summation being omitted for convenience. It follows that the suffixes a and b in the above term are dummies and may be replaced freely by any other letter not occurring in the same term, e.g.

$$A_{ab}{}^{iab} \equiv A_{jk}{}^{ijk}$$

but $A_{bb}{}^{ibb}$ is not a tensor. Apart from repeated suffixes, all terms in a tensor equation must carry the same suffixes in a similar position, for example the equation

$$A^{ij} = B^{ik}C_k{}^j + M_{ab}{}^{ia}N^{jb},$$

is legitimate. These rules extend to the partial differential coefficients in a transormation, the coefficient of the variable in the numerator of a partial differential oefficient being treated as an upper suffix, for example

$$\bar{A}_{jk}{}^{i} = \frac{\partial \bar{x}_i}{\partial x_a}\frac{\partial x_b}{\partial \bar{x}_j}\frac{\partial x_c}{\partial \bar{x}_k} A_{bc}{}^{a}$$

nd the summation convention still holds.

A vector is represented when necessary by the symbol **A** and its tensor components by A^i, A_i. Expressions containing commas among the suffixes, such as $A_{i,j}$, $A^i{}_{,a}$, $A_i{}^{,a}$, $A_{i,j,k}$, etc., stand for tensor derivatives as defined in § 6 and Appendix C.

Appendix B. *Additional properties of epsilon tensors.*

(i) If $A_k{}^{ij}$ is any tensor of the kind indicated, then

$$A_a{}^{ij}E_{jbc} + A_b{}^{ij}E_{jca} + A_c{}^{ij}E_{jab} = A_j{}^{ij}E_{abc} \qquad \text{......(B·1).}$$

Proof. If a, b, c are all different, both sides are zero. If, say, $b = c$, right hand side is zero and left hand side is the sum of terms of the type

$$(A_b{}^{ij}E_{jba} + A_b{}^{ij}E_{jab}) = 0.$$

If a, b, c are all different and equal to α, β, γ, which are fixed, i.e. they represent the numbers 1, 2, 3, in some order, then left hand side is equal to

$$A_\alpha{}^{i\alpha}E_{\alpha\beta\gamma} + A_\beta{}^{i\beta}E_{\beta\gamma\alpha} + A_\gamma{}^{i\gamma}E_{\gamma\alpha\beta}, \text{ not summed,}$$
$$= A_j{}^{ij}E_{abc}, \text{ summed.}$$

(ii) If A^{jk} and B^{abc} are arbitrary vectors

$$\{E_{ija}E_{kbc} + E_{ijb}E_{kca} + E_{ijc}E_{kab}\}\, A^{jk}B^{abc} = E_{ijk}A^{jk} \times E_{abc}B^{abc}$$
$$= \tfrac{1}{2}E_{ijk}\,(A^{jk} - A^{kj}) \times E_{abc}B^{abc} \qquad \text{......(B·2).}$$

Proof. Write left hand side $\equiv C \equiv C_1 + C_2 + C_3$. Since both sides vanish if $i = j$, we may put $i = \alpha$, $j = \beta$ where α, β, γ are fixed and all different. Hence $C_1 = 0$ unless $a = \gamma$. For $C_1 \neq 0$, $E_{kbc} \neq 0$: if $b = \gamma = a$, $C_3 = 0$, $C_2 = -C_1$, $C = 0$; if $c = \gamma = a$, $C_2 = 0$, $C_3 = -C_1$, $C = 0$; if $k = \gamma = a$,

$$C = E_{ijk}\,A^{jk}E_{abc}\,B^{abc}$$

restricted by the condition $a = \gamma$. Similarly for $C_2 \neq 0$, $C_3 \neq 0$, C has the above value restricted by the conditions $b = \gamma$, $c = \gamma$, respectively. Hence the total value of C is given by

$$C = E_{ijk}\,A^{jk}.E_{abc}\,B^{abc}$$

unrestricted,

$$= \tfrac{1}{2}\,E_{ijk}\,(A^{jk} - A^{kj}).E_{abc}\,B^{abc}.$$

Appendix C. *Covariant derivatives.*

It is required to find expressions for the tensors $A^i{}_{,a}$, $A_{i,a}$ defined by the equation

$$\delta A^i = A^i{}_{,a}dx^a \qquad \text{......(C·1),}$$
$$\delta A_i = A_{i,a}dx^a \qquad \text{......(C·2),}$$

where A^i, A_i are components of the vector **A** as defined in § 2. Let $\bar{A}^a = \bar{A}_a$ be components of the vector in the Cartesian system $\bar{X}_a$. Then

$$\delta\bar{A}_a = \frac{\partial \bar{A}_a}{\partial \bar{X}_b}\, d\bar{X}^b$$

and therefore

$$\delta A_i = \frac{\partial \bar{X}_a}{\partial x_i} \cdot \frac{\partial \bar{A}_a}{\partial \bar{X}_b} \cdot \frac{\partial \bar{X}_b}{\partial x_c}\, dx^c \qquad \text{......(C·3),}$$

and

$$\frac{\partial A_i}{\partial x_c} = \frac{\partial}{\partial x_c}\left\{\frac{\partial \bar{X}_b}{\partial x_i}\, \bar{A}_b\right\}$$

$$= \frac{\partial \bar{X}_a}{\partial x_c} \cdot \frac{\partial \bar{X}_b}{\partial x_i} \cdot \frac{\partial \bar{A}_b}{\partial \bar{X}_a} + \frac{\partial^2 \bar{X}_b}{\partial x_i \partial x_c} \cdot \frac{\partial x_r}{\partial \bar{X}_b} \cdot A_r \qquad \text{......(C·4).}$$

Comparing 3 and 4, we have

$$\delta A_i = \left\{\frac{\partial A_i}{\partial x_c} - A_r \frac{\partial x_r}{\partial \bar{X}_b} \cdot \frac{\partial^2 \bar{X}_b}{\partial x_i \partial x_c}\right\} dx^c \qquad \text{......(C·5),}$$

or

$$A_{i,c} = \frac{\partial A_i}{\partial x_c} - \Gamma_{ic}{}^{r} A_r \qquad \text{......(C·6),}$$

where

$$\Gamma_{ic}{}^{r} = \frac{\partial x_r}{\partial \bar{X}_b} \cdot \frac{\partial^2 \bar{X}_b}{\partial x_i \partial x_c} \qquad \text{......(C·7).}$$

To express $\Gamma_{bc}{}^{a}$ in terms of the derivatives of the g, we have, from the definition in equation (2·2),

$$\frac{\partial g_{ij}}{\partial x_k} = \frac{\partial \bar{X}_b}{\partial x_j} \cdot \frac{\partial^2 \bar{X}^b}{\partial x_i \partial x_k} + \frac{\partial \bar{X}_b}{\partial x_i} \cdot \frac{\partial^2 \bar{X}^b}{\partial x_j \partial x_k} \qquad \text{......(C·8).}$$

Write down the other two equations obtainable by interchanging ijk, add two together and subtract the third, and we get

$$\Gamma_{ij;k} \equiv \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x_i} + \frac{\partial g_{ki}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_k}\right) = \frac{\partial \bar{X}_b}{\partial x_k} \cdot \frac{\partial^2 \bar{X}^b}{\partial x_i \partial x_j} \qquad \text{......(C·9).}$$

This expression is called a Christoffel symbol of the first kind. Multiply equation (C·7) by

$$g_{sr} = \frac{\partial \bar{X}^b}{\partial x^r} \cdot \frac{\partial \bar{X}_b}{\partial x_s}$$

and we get

$$g_{sr}\, \Gamma_{ic}{}^{r} = \frac{\partial \bar{X}_b}{\partial x_s} \cdot \frac{\partial^2 \bar{X}^b}{\partial x_i \partial x_c} = \Gamma_{ic;s}.$$

It follows that

$$\Gamma_{ij}{}^{r} = g^{rk}\, \Gamma_{ij;k}$$

$$= \tfrac{1}{2} g^{rk}\left(\frac{\partial g_{jk}}{\partial x_i} + \frac{\partial g_{ki}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_k}\right).$$

Similarly it may be proved that

$$A^i{}_{,a} = \frac{\partial A^i}{\partial x_a} + \Gamma_{ra}{}^{i} A^r.$$

Appendix D. *Proof of the curl curl identity.*

This takes the form

$$(\text{curl curl } A)_i = \frac{\partial}{\partial x_i}\,(A^a{}_{,a}) - A_i{}^{,a}{}_{,a}$$

or

$$\text{curl curl } \mathbf{A} = \text{grad. div. } \mathbf{A} - \nabla^2 \mathbf{A}.$$

From equation (6·8)

$$(\text{curl } A)_i = g_{ia} E^{abc} A_{c,b},$$

$$\begin{aligned}(\text{curl curl } A)_i &= g_{ia} g_{ck} E^{abc} E^{kst} A_{t,s,b} \\ &= g^{br} E_{irk} E^{kst} A_{t,s,b} \\ &= g^{br} (\delta_i^s \delta_r^t - \delta_r^s \delta_i^t) A_{t,s,b},\end{aligned}$$

by equation (5·3)

$$\begin{aligned}&= g^{br} A_{r,i,b} - g^{br} A_{i,r,b} \\ &= \frac{\partial}{\partial x_i} (A^b{}_{,b}) - A_{i,}{}^{a}{}_{,a},\end{aligned}$$

the order of differentiation being assumed to be indifferent.

Appendix E. *Second method of simplifying the last term in the equation of viscous flow.*

The following method does not involve the use of the epsilon tensors, and is therefore more direct than that previously used. The resulting term in general orthogonal coordinates is more symmetrical than in equation (10·8) but is also considerably more complicated. The method is most simply applied to the covariant components, the term to be evaluated being $g^{bc} u_{i,b,c}$.

We have in full

$$u_{a,b} = \frac{\partial u_a}{\partial x^b} - \Gamma_{ab}{}^r u_r$$

and

$$A_{ab,c} = \frac{\partial A_{ab}}{\partial x^c} - \Gamma_{ac}{}^r A_{rb} - \Gamma_{bc}{}^r A_{ar}.$$

Hence, putting $A_{ab} = u_{a,b}$

$$\begin{aligned}u_{a,b,c} = \frac{\partial^2 u_a}{\partial x^b \partial x^c} &- \Gamma_{ac}{}^r \frac{\partial u_r}{\partial x^b} - \Gamma_{bc}{}^r \frac{\partial u_a}{\partial x^r} - \Gamma_{ab}{}^r \frac{\partial u_r}{\partial x^c} \\ &- u_s \frac{\partial \Gamma_{ab}{}^s}{\partial x^c} + \Gamma_{ac}{}^r \Gamma_{rb}{}^s u_s + \Gamma_{bc}{}^r \Gamma_{ar}{}^s u_s \quad \text{......(E·1)}.\end{aligned}$$

Hence

$$g^{bc} u_{a,b,c} = g^{bc} \frac{\partial^2 u_a}{\partial x^b \partial x^c} - g^{bc} \Gamma_{bc}{}^r \frac{\partial u_a}{\partial x^r} - 2 g^{bc} \Gamma_{ac}{}^r \frac{\partial u_r}{\partial x^b} + \text{terms in } u_s \quad \text{...(E·2)}.$$

The first two terms may be combined by analogy with $g^{bc} \left(\frac{\partial \phi}{\partial x^b}\right)_{,c}$ which may be written in the alternative forms

$$g^{bc} \frac{\partial^2 \phi}{\partial x^b \partial x^c} - g^{bc} \Gamma_{bc}{}^r \frac{\partial \phi}{\partial x^r}$$

and

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^c} \left(g^{bc} \sqrt{g} \frac{\partial \phi}{\partial x^b} \right).$$

Hence

$$g^{bc} u_{a,b,c} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^c} \left(g^{bc} \sqrt{g} \frac{\partial u_a}{\partial x^b} \right) - 2 g^{bc} \Gamma_{ac}{}^r \frac{\partial u_r}{\partial x^b} + \text{terms in } u_s.$$

In Euclidean space the terms in u_b (terms of zero order in u_a) may be simplified by the following artifice. It appears that the terms of zero order in equation (E·1) are unaltered by interchanging a and b and hence $u_{a,b,c} - u_{b,a,c}$ contains no terms of zero order in u_a. Now

$$g^{bc}u_{a,b,c} - \frac{\partial}{\partial x^a}(g^{bc}u_{b,c}) = g^{bc}(u_{a,b,c} - u_{b,c,a})$$
$$= g^{bc}(u_{a,b,c} - u_{b,a,c} + u_{b,a,c} - u_{b,c,a}),$$

and the last two terms cancel since the order of differentiation is indifferent in Euclidean space. Hence

$$g^{bc}u_{a,b,c} = \frac{\partial}{\partial x^a}\{g^{bc}u_{b,c}\} + \text{terms of 1st and 2nd order}$$
$$= \frac{\partial}{\partial x^a}\left\{\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^c}g^{bc}\sqrt{g}u_b\right\} + \text{terms of 1st and 2nd order}$$
$$= u_b\frac{\partial}{\partial x^a}\left\{\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^c}(g^{bc}\sqrt{g})\right\} + \text{terms of 1st and 2nd order.}$$

The first and second order terms have already been evaluated and we have finally

$$g^{bc}u_{a,b,c} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^c}\left(g^{bc}\sqrt{g}\frac{\partial u_a}{\partial x^b}\right) - 2g^{bc}\Gamma_{ac}^{r}\frac{\partial u_r}{\partial x^b} + u_b\frac{\partial}{\partial x^a}\cdot\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^c}(g^{bc}\sqrt{g}),$$

or, writing the second term in full,

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^c}\left(g^{bc}\sqrt{g}\frac{\partial u_a}{\partial x^b}\right) - g^{bc}g^{rs}\frac{\partial u_r}{\partial x^b}\left(\frac{\partial g_{as}}{\partial x^c} + \frac{\partial g_{cs}}{\partial x^a} - \frac{\partial g_{ac}}{\partial x^s}\right) + u_b\frac{\partial}{\partial x^a}\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^c}(g^{bc}\sqrt{g})$$
$$\text{......(E·3).}$$

In general orthogonal coordinates in the notation already used this reduces to the following form

$$\frac{1}{a} \times \text{first component of } (g^{bc}u_{i,b,c})$$
$$= \frac{1}{a^2bc}\left\{\frac{\partial}{\partial x}\frac{bc}{a}\cdot\frac{\partial aU}{\partial x} + \frac{\partial}{\partial y}\frac{ca}{b}\cdot\frac{\partial aU}{\partial y} + \frac{\partial}{\partial z}\frac{ab}{c}\cdot\frac{\partial aU}{\partial z}\right\}$$
$$- \frac{2}{a^2}\left\{\frac{1}{a^2}\frac{\partial a}{\partial x}\cdot\frac{\partial aU}{\partial x} + \frac{1}{b^2}\frac{\partial a}{\partial y}\cdot\frac{\partial aU}{\partial y} + \frac{1}{c^2}\frac{\partial a}{\partial z}\cdot\frac{\partial aU}{\partial z}\right\}$$
$$- \frac{2}{a}\left\{\frac{1}{a^3}\frac{\partial a}{\partial x}\cdot\frac{\partial aU}{\partial x} + \frac{1}{ab^2}\frac{\partial a}{\partial y}\cdot\frac{\partial bV}{\partial x} + \frac{1}{ac^2}\frac{\partial a}{\partial z}\cdot\frac{\partial cW}{\partial x}\right\}$$
$$- \frac{2}{a}\left\{\frac{1}{a^3}\frac{\partial a}{\partial x}\cdot\frac{\partial aU}{\partial x} + \frac{1}{b^3}\frac{\partial b}{\partial x}\cdot\frac{\partial bV}{\partial y} + \frac{1}{c^3}\frac{\partial c}{\partial x}\cdot\frac{\partial cW}{\partial z}\right\}$$
$$+ \frac{1}{a}\left\{aU\frac{\partial}{\partial x}\frac{1}{abc}\frac{\partial}{\partial x}\left(\frac{bc}{a}\right) + bV\frac{\partial}{\partial x}\frac{1}{abc}\frac{\partial}{\partial y}\left(\frac{ca}{b}\right) + cW\frac{\partial}{\partial x}\frac{1}{abc}\frac{\partial}{\partial z}\left(\frac{ab}{c}\right)\right\} \quad \text{......(E·4),}$$

each line of this equation corresponding to a term of equation (E·3). The sum of the first and second lines may also be written

$$\frac{1}{abc}\left\{\frac{\partial}{\partial x}\frac{bc}{a}\frac{\partial U}{\partial x}+\frac{\partial}{\partial y}\frac{ca}{b}\frac{\partial U}{\partial y}+\frac{\partial}{\partial z}\frac{ab}{c}\frac{\partial U}{\partial z}\right\}$$

$$aU\left\{\frac{1}{a^2}\frac{\partial}{\partial x}\left(\frac{1}{a^2}\frac{\partial a}{\partial x}\right)+\frac{1}{b^2}\frac{\partial}{\partial y}\left(\frac{1}{a^2}\frac{\partial a}{\partial y}\right)+\frac{1}{c^2}\frac{\partial}{\partial z}\left(\frac{1}{a^2}\frac{\partial a}{\partial z}\right)\right\}$$

$$\frac{U}{a^2bc}\left\{\frac{\partial a}{\partial x}\frac{\partial}{\partial x}\left(\frac{bc}{a}\right)+\frac{\partial a}{\partial y}\frac{\partial}{\partial y}\left(\frac{ca}{b}\right)+\frac{\partial a}{\partial z}\frac{\partial}{\partial z}\left(\frac{ab}{c}\right)\right\}.$$

It may be verified by direct substitution that these forms are equivalent to that included in equation (10·8).

Appendix F. *Method of reducing the term* $A^i \equiv \{\mathrm{curl}\,(u^j{}_{,b}u^b)\}^i \equiv E^{iab}\,(u_{b,c}u^c)_{,a}$. We have

$$\begin{aligned} 0 &= \tfrac{1}{2}\,\{\text{curl grad.}\,(u^b u_b)\}^i \\ &= \tfrac{1}{2}E^{ica}\,(u^b u_b)_{,a,c} \\ &= E^{ica}\,(u^b u_{b,a})_{,c} \\ &= E^{ica}\,\{u^b\,(u_{a,b}+E_{abk}\xi^k)\}_{,c}, \end{aligned}$$

since $u_{b,a}-u_{a,b}=E_{abk}\xi^k$ by equations (5·6), 6·8), (7·12), or

$$0 = A^i + (\delta^i_b\delta^c_k - \delta^i_k\delta^c_b)\,(u^b\xi^k)_{,c},$$

using equation (5·3); or

$$A^i = (\xi^i u^c)_{,c} - (u^i\xi^c)_{,c}.$$

Now

$$\xi^i{}_{,c}u^c - u^i{}_{,c}\xi^c = u^c\frac{\partial\xi^i}{\partial x_c} - \xi^c\frac{\partial u^i}{\partial x_c},$$

since the terms containing Christoffel symbols cancel. Also $\xi^b{}_{,b}$ vanishes identically. Hence we have finally

$$A^i \equiv \{\mathrm{curl}\,(u^j{}_{,b}u^b)\}^i = u^a\frac{\partial\xi^i}{\partial x_a} - \xi^a\frac{\partial u^i}{\partial x_a} + u^a{}_{,a}\xi^i.$$

DISCUSSION

Prof. W. WILSON. Mr Lock has succeeded in showing that the utility of the tensor calculus is not restricted to relativistic investigations and that it can be applied with advantage to hydrodynamical and similar three-dimensional problems in Euclidean space. This is due to the fact that the equations in these problems are, for the most part, statements of the equality of different expressions for the same component of some vector or tensor, and these equations retain their validity when the coordinate system is changed. It is easy to see why the tensor calculus developed in association with the theory of relativity. The most important principle of that theory is that the equations describing physical phenomena have the same form whatever may be the system of reference. I should like to congratulate Mr Lock on his very interesting paper.

Mr T. Smith commended tensor methods, and all algebraic methods in which a single symbol denotes a number of terms, to the attention of physicists, and said that they not only simplify the formulation of problems but also lend themselves to very rapid numerical calculation. The distinction between covariants and contravariants is important; physicists interested in a particular case, as distinct from a more general case, are sometimes misled by the similarity between the coefficients in covariant and contravariant expressions and consequently reach a wrong result He thought the paper would be improved if the author would explain some points in the notation, such as the use of a comma to denote differentiation.

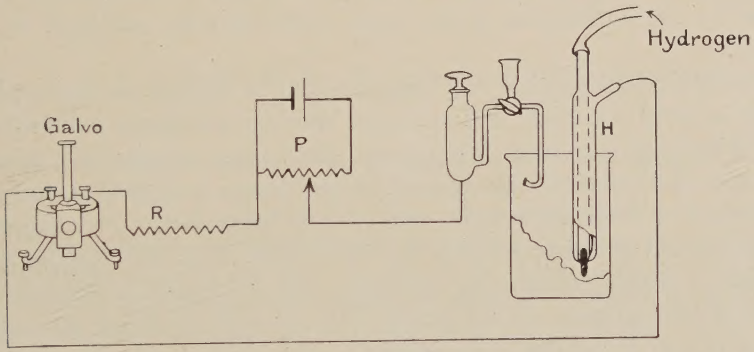
Capt. C. W. Hume. Is it essential to use commas to denote differentiations? Formulae printed in this way are rather confusing, especially when an equation is followed by a punctuational comma.

Author's reply. A good example of the necessity for distinguishing between covariant and contravariant components is the case of the symbol ∇^2 applied to a vector and to a scalar, where, as remarked in § 6, the actual formulae, while identical in Cartesians, are different in curvilinear coordinates. A third different expression is that denoted by $D(\psi)$ in equation (12·5) for the case of motion symmetrical about an axis. Modifications have been made to the paper in proof to explain more clearly the use of the comma notation. In answer to Capt. Hume, the comma, as denoting covariant differentiation, appears the most convenient of the notations one has seen, and, if it is suggested that the comma might escape notice, it may be remarked that Eddington, in his original tracton relativity, used no special notation at all, writing A_{ab} indifferently for a tensor of the second order or for the covariant derivative of the vector A_a. The ordinary differential notation cannot be used, since the covariant derivative is not identical with the ordinary partial derivative.

DEMONSTRATIONS

A *p*H Apparatus. *Demonstration given on January* 24, 1930, *by* W. E. DORAN, M.A., The Cambridge Instrument Co., Ltd.

A solution is either acid, alkaline or neutral, and the degree of effective acidity or alkalinity is entirely dependent upon the concentration of undissociated hydrogen-ions in the solution. This hydrogen-ion concentration, which may be represented in *p*H units, can be accurately determined by immersing a hydrogen electrode in the solution and measuring the electric potential set up. The outfit demonstrated is designed for this purpose, and gives direct readings in *p*H units on the scale. It consists of a hydrogen electrode, calomel electrode, portable potentiometer, reflecting galvanometer and scale, together with a supply of hydrogen. The potentiometer is employed both for standardising the galvanometer deflection and also for balancing out the calomel e.m.f., the deflection of the galvanometer then being due to the e.m.f., of the hydrogen electrode only.



C = Calomel electrode; H = Hydrogen; P = Potentiometer; R = High resistance.

At the beginning of the demonstration, the hydrogen electrode was immersed in a solution which was neither acid nor alkaline, and the scale reading was 7, which is the *p*H value of a neutral solution. By the addition of a drop of nitric acid the *p*H reading was reduced, indicating an increase in acidity, and with a little sodium hydroxide the *p*H value was increased until at 7 it was again neutral. A further addition of sodium hydroxide caused the liquid to assume an alkaline state. In this way it was demonstrated that the acidity of the solution can easily be varied and controlled to almost any desired value by means of the apparatus.

The electrode method of measuring hydrogen-ion concentration is now used in nearly every industrial process. As an example of its value, mention was made of the importance of measuring the acidity of beer during manufacture. If the beer is too acid the effects may be detrimental to the consumer and at least unpleasant to the palate; on the other hand, if the acidity is too low it is favourable to the growth of bacteria. The actual *p*H value varies in different types of beer. Among other applications may be mentioned the tanning industry, sugar manufacture, bread making, preserved food manufacture, textile dyeing—in fact, there are few industrial processes in which hydrogen-ion concentration does not play an important part.

Model illustrating the Mosaic Theory of the Compound Eye, *lent by* The Department of Zoology, University of London, King's College. *Demonstration given on February* 14, 1930, *by* Capt. C. W. HUME.

The eye of an insect consists of a large number of tubes (ommatidia) formed with small lenses at their outer ends and associated with a retina at their inner ends, the number of ommatidia varying from 10,000 upwards. It has been suggested that each lens forms an independent image, the effects of all the images being combined as in the case of binocular vision. According to Alternburg*, however, the lenses serve merely as condensers. Light from any point of the object can pass through that one of the ommatidia which is pointing directly at it, but is absorbed by the walls of the other ommatidia. An erect image is consequently formed on the retina by the collocation of the small light-elements admitted by different ommatidia.

Altenburg's model consists of a large number of contiguous straws, blackened to prevent internal reflection, and arranged between a brightly coloured and illuminated object and a ground-glass screen. Each straw admits light from a small element of the field of view, and a faint but clear image is formed on the screen.

In the case of the insect eye the axes of the ommatidia are convergent towards the retina, instead of being parallel like the straws. Thus the angular magnitude of the object is always the same for an insect, and the definition decreases in direct proportion to the distance of the object. The amount of detail given by an eye containing 10,000 ommatidia is equivalent to that given in television by a Nipkow disc with 100 apertures, whereas the number of apertures usually employed is about 30.

The illumination of the insect retina is weak, as only a very narrow pencil is admitted from each point on the object. The retina is probably more sensitive than the retina of a vertebrate, but the weakness of illumination may account for the inactivity of insects on cloudy days.

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* *British Journal of Experimental Biology*, **4**, 38 (1926).

REVIEWS OF BOOKS

Cambridge Five-figure Tables, by F. G. HALL, M.A. and E. K. RIDEAL, M.A. Pp. viii + 76. (London: Cambridge University Press.) 3*s*. 6*d*.

To the making of books of tables there is no end, but there is still room for a compact book of five-figure tables, published at a reasonable price. The book before us, containing logarithms, natural sines, circular measure, values of e^x and e^{-x}, useful constants, atomic weights, and a list of "useful differentials and integrals," is full but compact, and can be commended. The type is not all that could be desired, but the page is saved from overcrowding by the compilers' practice of breaking it up into short columns and, after the first five-figure entry, printing, in any one column, only the last three figures. No difference columns are necessary. The reviewer confesses to a hankering after the type used in Barlow's tables.

The table of constants is useful and might with advantage be extended, and the list of differential coefficients and integrals will bring joy to the heart of many a toiling chemist.

A. F.

Magnetism, by E. C. STONER, Ph.D. Pp. vii + 114. (London: Methuen & Co., Ltd.) 2*s*. 6*d*.

Dr Stoner has produced a small but very useful work on the magnetic properties of materials. He deals briefly but capably with the magnetic properties of atoms, diamagnetism, paramagnetism, ferromagnetism, and the magnetic properties of the elements. His book is confessedly one for those to whom the subject is not new, and perhaps plunges *in medias res* a little suddenly for those who like to be dosed "with learning put lightly, like powder in jam," but, as the author hopes, it is admirably fitted for those whose desire it is to reach a stage at which they may carry out research on the subject. It can be recommended strongly.

A. F.

Magnetic, Meteorological and Seismographic Observations made at Bombay and Alibag, 1924, under the direction of S. K. Banerji, D.Sc. Pp. iii + 133. (Calcutta: Government of India Publication Branch.) 11*s*.

The delay in the appearance of this volume is of little moment in view of the great advance that it marks, an advance on which India is warmly to be congratulated. The equipment and programme of the Bombay Observatory was largely modelled on that of Kew, but those whose work required hourly values of any of the elements would look in vain for them in the first forty-two annual volumes. The breaking down of tradition caused by the war led to the publication of two-hourly values of all the important elements in 1921, though the cost of printing hourly values was then regarded as prohibitive: but thanks to support from the Royal Society the hourly scheme has now been adopted for the data of pressure, wind, temperature, relative humidity, cloud, the magnetic elements and microseisms. No very abnormal winds are recorded, the greatest velocity being only 27 miles an hour.

G. T. W.

Greenwich Magnetic and Meteorological Observations. Pp. D 62 + E 46. (London: H.M. Stationery Office, 1927.) 7*s*. 6*d*.

The electrification of railways forced the magnetic observatory to be moved from Greenwich to near Abinger on the northern slope of Leith Hill, and the present equipment includes a Schuster-Smith coil magnetometer for absolute measurements of horizontal force as well as an earth inductor for absolute inclination. The meteorological instruments still include a black-bulb radiation thermometer, in spite of the difficulties elsewhere found with changes in the sensitiveness of this instrument. G. T. W.

The Observatories' Year Book, 1927. Pp. 430. (London: H.M. Stationery Office.) £3. 3*s*.

This volume contains not only data of continuously recorded terrestrial magnetism from Lerwick and Eskdalemuir, of atmospheric electricity from Lerwick, Eskdalemuir and Kew, and of meteorology from Aberdeen, Eskdalemuir, Cahirciveen (Valentia) and Kew; but also of auroras from Lerwick, and of underground water-level, atmospheric pollution and seismology from Kew; and there is an aerological section with upper-air data derived from registering-balloons.

With regard to the microseisms, which are presumably due to waves at sea and so have about seven times the amplitude in winter that they have in summer, it would be interesting to examine whether very long waves of small amplitude could be used as a warning of the existence of distant storms.

India is now playing her part in the publication of hourly values of her data: we hope that other portions of the Empire may see their way to make their geophysical materials more readily available for general use. G. T. W.